Math for Poets and Drummers

The Mathematics of Meter

Rachel Hall
Saint Joseph's University
5600 City Avenue
Philadelphia, PA 19131
rhall@sju.edu

September 27, 2005

Music has many connections to mathematics. The Greeks discovered that chords with a pleasing sound are related to simple ratios of integers. It is less well known that the rhythms of music and poetry have been studied mathematically—in fact, Indian scholars first discovered Pascal's triangle and the Fibonacci numbers in the rhythms of poetry. This article explores some mathematics related to rhythm in poetry and music.

In English and Spanish, the rhythm, or meter, of a piece of poetry is determined by its pattern of stressed and unstressed syllables. For example, the works of Shakespeare are written in the meter called iambic pentameter: five pairs of alternating unstressed and stressed syllables to a line of poetry. In Sanskrit, the classical language of India, meter is based on length, rather than accent; syllables in Sanskrit are either short or long, and meters describe the pattern of short and long syllables. Some authors list over 100 meters. Perhaps because of this wealth of meters, at least two ancient scholars investigated the question "how can we count the possible patterns of short and long syllables?"

The Sanskrit scholar Pingala (c. 200 BC) counted the possible poetic meters of a fixed number of syllables in his *Chandahsutra* (Bag, 1966). Since syllables are short or long, there are 2^n meters of n syllables. However, Pingala made a more interesting observation. Within the set of all patterns of n syllables, he counted patterns having the same number of short syllables. For example, he would classify the 16 different meters of four syllables in the following way:

1	meter of four short syllables (SSSS);
4	meters of three shorts (SSSL, SSLS, SLSS, LSSS);
6	meters of two shorts (SSLL, SLLS, LLSS, ...);
4	meters of one short (LLLS, LLSL, LSLL, SLLL);
1	meter of no shorts (LLLL).

He proceeds to describe the addition rule for finding the rows of Pascal's Triangle. Pingala did not seem aware of other applications of his discovery (for example, to problems like coin tossing).

The Jain writer Ācārya Hemacandra (c. 1150 AD) also studied poetic meter (Singh, 1986). Again, syllables are long or short; he set the length of a long syllable to be twice that of a short syllable. The length of a line of poetry is the sum of lengths of its syllables, counting short syllables as 1. Hemacandra counted meters in which the length of a line is fixed, but the number of syllables is not fixed. I ask my students to list and count the possible meters for each length, as below:

length	meters	number of meters
1	S	1
2	SS, L	2
3	SL, SSS, LS	3
4	SSL, LL, SLS, SSSS, LSS	5

Hemacandra (and my students) discovered that each number in the sequence on the right hand column is found by adding the two numbers above it. In other words, he found the Fibonacci numbers—half a century before Fibonacci! Singh theorizes that Fibonacci, who was educated in North Africa and came into contact with Eastern mathematics, might have been influenced by Indian knowledge of the sequence. In any case, the reason the Fibonacci sequence counts these patterns is as follows.

Let P_n be the number of patterns of length n. Partition these patterns into two groups: those that end with a short syllable and those that end with a long syllable. Each pattern ending with a short syllable is a string of syllables of total length $n-1$ followed by a short syllable; the number of strings of length $n-1$ is P_{n-1}, and therefore the number of patterns of length n that end with a short syllable is P_{n-1}. Each pattern ending with a long syllable begins with a string of syllables of total length $n-2$, followed by a long syllable. Therefore, there are P_{n-2} of these. Finally, P_n, the total number of patterns of length n, equals $P_{n-1}+P_{n-2}$. Since $P_1 = 1$ and $P_2 = 2$, we obtain the Fibonacci sequence.

This derivation of the Fibonacci numbers is identical to the domino/square problem: In how many ways can one tile a row of size $1 \times n$ units with dominoes of size 1×2 units and squares of side 1 unit? Several similar Fibonacci problems can be found on Ron Knott's web page (Knott, 2005). Figure 1 provides a visual explanation of the solution to the domino/square problem; it mirrors the argument in the previous paragraph.

The poetic meters that Pingala and Hemacandra studied have an analog in music: in many cultures, different types of dance music are identified with different repeated rhythm patterns. Rhythm patterns are formed by grouping beats into notes, which play the role of syllables in poetry. A drum is hit on the first beat of each note and silent on the following beats; the length of a note is the number of beats it occupies. For example, the *clave*, the repeated rhythm pattern played by the claves (wooden sticks that are hit together), is a characteristic of salsa music. The clave and many other characteristic rhythm patterns from around the world are composed of notes of length 1 and 2; one also finds many patterns consisting of notes of length 2 and 3. Figure 2 shows a few examples. To hear some of these patterns, I suggest SongTrellis (Luebbert, 2005), where you can hear the merengue and cumbia bell parts both separately and in context. The guajira may be familiar as the rhythm of Leonard Bernstein's "America," from West Side Story. I encourage the reader to experiment with creating and listening to rhythm patterns; a good place to start is the web applet Jas's MIDI Hand Drum Rhythm Generator (Senn, 2005). You can hear compositions by my students on my site http://www.sju.edu/∼rhall/Multi.

What sequence counts numbers of patterns consisting of two- and three-beat notes? Here are the first twelve entries of this sequence:

length (n)	1	2	3	4	5	6	7	8	9	10	11	12
number of patterns (R_n)	0	1	1	1	2	2	3	4	5	7	9	12

If R_n is the number of such patterns, then $R_n = R_{n-2} + R_{n-3}$. The proof of this statement is similar to the argument for notes of length one and two. In this case, break the patterns of length n into patterns of length $n-2$ followed by a 2-beat note and patterns of length $n-3$ followed by a 3-beat note.

Though not nearly as famous as the Fibonacci numbers, this sequence, named the Padovan numbers, has some interesting properties. It is well known that the limit of the ratios of successive Fibonacci numbers is the golden number; the limit of the ratios of successive Padovan numbers is the so-called "plastic number." To find this number, observe that

$$\frac{R_n}{R_{n-1}} = \frac{R_{n-2}}{R_{n-1}} + \frac{R_{n-3}}{R_{n-1}} = \frac{R_{n-2}}{R_{n-1}} + \frac{R_{n-3}}{R_{n-3} + R_{n-4}}$$

Take the limit as $n \to \infty$ of both sides and let $r = \lim_{n\to\infty} R_n/R_{n-1}$. Then

$$r = \frac{1}{r} + \frac{1}{1 + 1/r}$$

so r is a solution to the cubic equation $r^3 - r - 1 = 0$. The only real root of this equation is the plastic number $r = 1.3247179572447\ldots$. There are several interesting applications of the Padovan numbers—for example, they are related to a spiral of equilateral triangles in the way the Fibonacci numbers are related to a spiral of squares (see Figures 3). See Ian Stewart's article "Tales of a Neglected Number" for other ideas (Stewart, 2004).

References

Amulya Kumar Bag. Binomial theorem in ancient India. *Indian J. History Sci.*, 1: 68–74, 1966.

Ron Knott. Fibonacci numbers and the golden section, 2005. URL: http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html. Retrieved February 9, 2005.

David Luebbert. The rhythms on SongTrellis, 2005. URL: http://www.songtrellis.com/rhythmPage. Retrieved February 9, 2005.

Jeff Senn. Jas's – MIDI hand drum rhythm generator, 2005. URL: http://www.khafif.com/rhy/rhygen.html. Retrieved February 9, 2005.

Parmanand Singh. Ācāraya Hemacandra and the (so-called) Fibonacci numbers. *Math. Ed. (Siwan)*, 20(1):28–30, 1986. ISSN 0047-6269.

Ian Stewart. *Math Hysteria : Fun and games with mathematics*. Oxford University Press, Oxford, 2004.

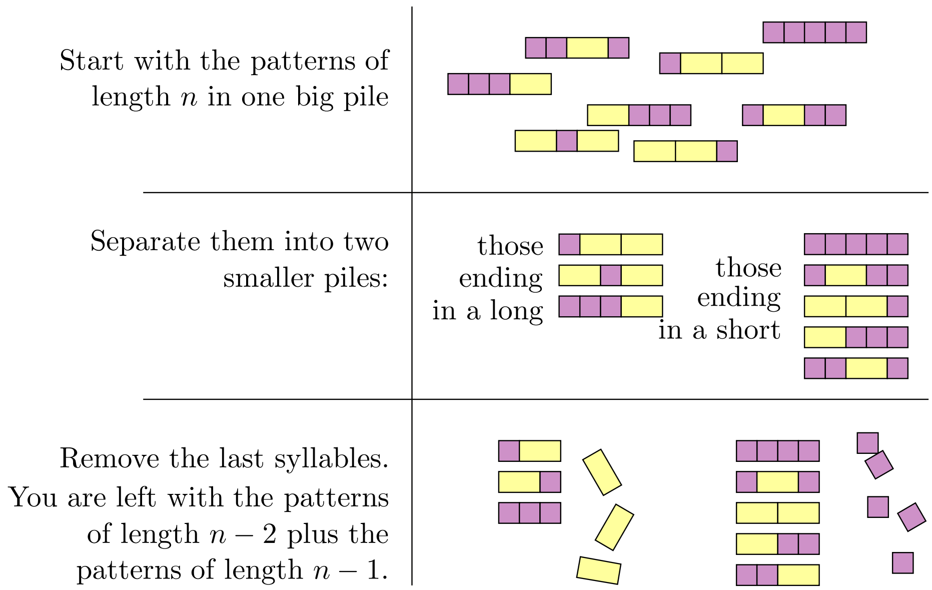

Figure 1: Visual solution to the domino/square problem

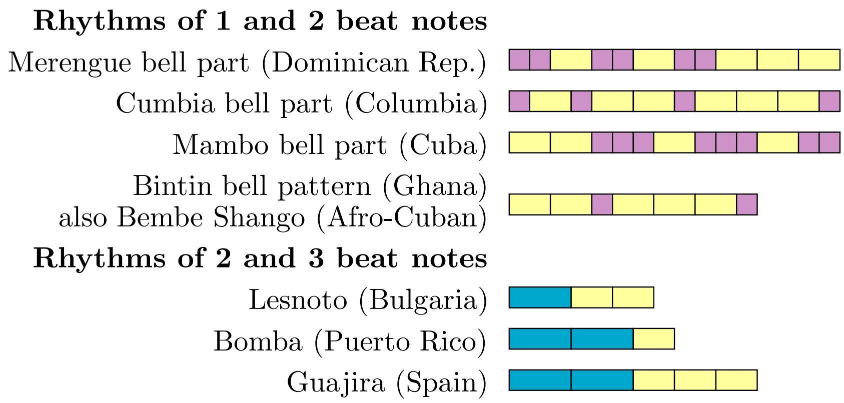

Figure 2: Dance rhythms

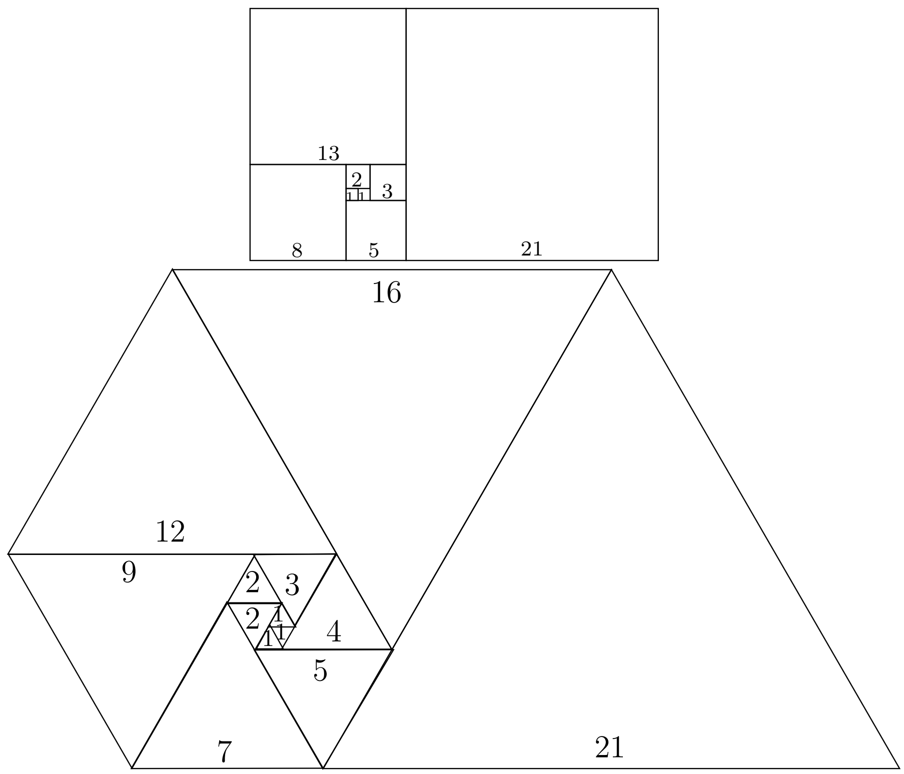

Figure 3: The Fibonacci numbers and Padovan numbers