Instructor Notes 8: Concavity

Purpose

This tool demonstrates the relationship between the function and its second derivative. That is, the tool shows students graphically the relationship between the concavity of the function and the graph of $f(x)$. The second derivative not only indicates for what values the function is concave upwards and concave downwards, but also the numerical value of $f''(x)$ is indicative of the change in the steepness of the curve. The tool demonstrates that the inflection point occurs when the rate of change switches from being an increasing function to a decreasing function.

Prerequisites

- Knowledge of the second derivative.
- The concepts of concavity and inflection point.

Concepts developed

In this tool, students will see the graph of $f(x)$ in the upper window and the graph of $f''(x)$ in the lower window. By studying the values of $f''(x)$ and comparing the graphs, the idea of concavity can be reinforced. Inflection points are clearly indicated on the graph [$f''(x) = 0$] where the graph crosses the $x -$ axis. The corresponding point is clearly indicated on the graph of $f(x)$.

The tangent lines are drawn on the graph of $f(x)$ so that the students can see that as the cursor rolls over the graph, if the slopes are increasing, the second derivative is positive. When the slopes are decreasing in value, the second derivative is negative.

Students are asked to observe that a function is *concave up* if the graph lies above the tangent lines and bends upward above the tangent lines. The function is *concave down* when the graph lies below the tangent lines, and bends downward from the tangent lines. Another characteristic to note is that the second derivative is positive means that the function representing the slopes of the tangent lines, f', is increasing.

The tools also uses parameters to represent constants. When the values of the parameters change, the students can note the effect on the shape of the function.

Tool Instructions

Open the **Concavity Tool** in the Derivative Kit, which wakes up with the graph of the function $f(t) = ate^{-bt}$, a surge-dissipation curve. This function is a model for several natural phenomenon. If you like you can think of this particular function as representing the sale of a "hot" new item. (Recall that the slope of the tangent lines indicate the rate.) In the lower graph is the function $f''(t)$. Using the sliders, choose values for a and b. Observe what happens to $f''(t)$ as you move left to right through the graph of $f(t)$. Record several values of a and b and the value of t for which $f''(t) = 0$ for each choice of a and b.

Sample classroom demonstration - Concavity

Using this tool there are several questions you might ask to have students verbalize and visualize the relationship between the function and its second derivative. In addition, you can use this to explore the relationship between parameters and the shape of the function. Discuss with the class when $f(t)$ is concave up and when $f(t)$ is concave down. Use the lower window, which contains the graph of $f''(t)$, to answer this question. Also use the lower graph to locate approximate value(s) of the inflection point(s). Ask students what information about $f(t)$ can be found from $f''(t)$. [You can cover the upper graph if you like.]

Some questions you might ask:

Questions concerning the opening function:

What is the effect of the number of sales on the future sales, if any? What role does the inflection point play?

What happens to the sales for $t < \frac{2}{b}$? For $t > \frac{2}{b}$

What are the effects of a and b on f'' and f?

Have students calculate f'' and explain the relationship among a, b and the inflection point.

Among the functions we have seen, the simplest are functions of the family $f(x) = x^n$. In this context, we are interested in the concavity of the graph of x^n for various values of n. Choose the function $f(x) = x^n$ to answer the following questions.

Using only positive values of x, the concavity of the function $f(x) = x^n$ depends on the value of n. Ask students to find $f''(x)$.

For what values of n is the function always concave up?

For what values of n is the function always concave down?

When the second derivative is negative, where are the tangent lines in relationship to the graph of the function? Similarly, when the second derivative is positive, where are the tangent lines in relationship to the graph of the function?

Ask students to describe what the numerical value of the second derivative tells you about the graph of the function.

Does the graph have any inflection points?

You may repeat this exercise with several of the other functions given in the table. Each of these functions models some physical phenomenon including several growth models, a model of heating an object, and a model of cooling an object to room temperature. The function which appears when one opens the **Tangent** tool is also present so that if you have used that tool for a demonstration, you may continue to develop more information about the graph of using the **Concavity** tool.