

Lab 19: Partial Derivatives

Economics is the study of such complex phenomena that functions of several independent variables must be used to model them. The trick to analyzing these functions is to hold all the variables fixed except one and find the derivative of the resulting function. We'll look at the Cobb-Douglas production function to see how this process works and what we can learn from it.

Open the **Partial Derivatives** tool in the **Functions of Two Variables Kit**.

Note that you obtain a surface $z = f(x, y)$ for which the vertical distance z represents the value of the function for given values of x and y . The surface displayed is the graph of a function of special interest in economics:

1. Introducing the Cobb-Douglas Function

The Cobb-Douglas function was invented in the 1920's to model the production P of the U.S. economy when the total labor costs x and total capital investment y are known. Later it has had other uses, modifications and extensions.

In this instance we will apply it to an industry for which there are x units of labor costs and y units of capital expenditure (on machinery, automation, computers, and facilities, etc.) Both x and y are independent variables. The total production P resulting from these variable inputs is the **Cobb-Douglas Production function** for some parameter c , where $0 < c < 1$:

$$P = f(x, y) = K x^c y^{1-c} . \quad (1)$$

(For convenience on the graph, we have chosen $K = 1$.)

- 1.1 Show that the Cobb-Douglas function given in (1) has the property that if we double the labor input x and double the capital expenditure, then resulting production P is doubled as well. (Hint: replace x by $2x$ and y by $2y$ in equation (1)).

- 1.2 What happens if we triple both x and y ? What happens if we try nx and ny in (1) instead?

This property is called **constant returns to scale**.

2. The Partial Derivative

When we begin to analyze a function of two variables, we would like to determine rates of change with respect to each independent variable. The method for doing this is to handle the case for each variable **separately**.

The **partial derivative of $f(x, y)$ with respect to x** , written as $f_x(x, y)$, is the ordinary derivative of $f(x, y)$ with y treated as a constant. Similarly, the **partial derivative of $f(x, y)$ with respect to y** , written $f_y(x, y)$, is the ordinary derivative of $f(x, y)$ with x treated as a constant. Frequently these partial derivatives are written as:

$$f_x(x, y) = \frac{\partial f}{\partial x} \text{ and } f_y(x, y) = \frac{\partial f}{\partial y}.$$

Now let's use the tool to see what is happening geometrically as we compute these partial derivatives. Use the default value for c so that $c=0.5$.

To use the tool:

- Set the parameter c by clicking the cursor on the *c-slider*.
- Look at the window in the lower right. The function pictured corresponds to the function $z = f(x, b)$ where b is fixed. (The default value for $b=1$). Look for the slice in the upper window that corresponds to the intersection of the plane $y = b$ with the surface $z = f(x, y)$. Roll the cursor over the lower window to see the tangent line move on the slice and on the surface. The slope of that tangent line is $f_x(x, b)$.
- Now roll the cursor over the left lower window. The slope of the tangent line obtained is $f_y(a, y)$.

- To change the selection for the fixed x -value a , simply click the cursor on at the a -value desired in the lower right window. In similar fashion, you can change the fixed y -value b by clicking the cursor on the lower left screen.

3. Interpretation of the Partial Derivative as Marginal Productivity

- 3.1 Set the parameter $c=0.4$ and assume that the capital expenditures are fixed at $y = b = 1$. Let x represent the units of labor applied. Assume that x is number of workers in thousands and y is the cost of capital expenditures in millions of dollars. (The constant K in (1) is a constant of proportionality that gives the production in millions of units.) Increase x by rolling the cursor over the graph in the lower right window. Describe what happens to the *slope* of the tangent line as x is increased. (The slope of the tangent line is $f_x(x, b)$ and is called the marginal productivity of labor at (x, b) .)
- 3.2 For the Cobb-Douglas function, note from the tool that the marginal productivity of labor is always decreasing. What does that mean in terms of concavity? (Check your results for several values of b .)

- 3.3 In economic terms, compare the increase in production if you hold the capital expenditure fixed and you add more workers at $x = 1$ to what you would get if you add more workers at $x = 3$. When is it more effective to add workers (or perhaps to pay overtime)?
- 3.3 Do exercise 3.1 and 3.2 using $f_y(a, y)$ instead. Do you get similar results for $f_y(a, y)$?
- 3.4 The slope of the tangent line is $f_y(a, y)$ and is called the marginal productivity of capital at (a, y) . Is the **marginal productivity of capital** increasing or decreasing as y increases? (Check your results for several values for a .)
- 3.5 In economic terms compare the increase in production if the number x of workers (in thousands) is held constant and new capital expenditure (such as computers or machinery) are added when $y=1$ with that when the same capital expenditure is added when $y=3$. When is it more effective to add capital expenditures?

3.6 Use equation (1) and determine the partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Remember that the parameter c is a constant. Also recall that for $f_x(x, y)$, the quantity y is treated as a constant and for $f_y(x, y)$, the quantity x is treated as a constant.

3.7 Set $c = 4$. Evaluate $f_x(1, 2)$ and $f_y(1, 2)$. Show your work and check your results on the tool.

4. Conclusion

The Cobb Douglas production function is particularly useful because it is continuous and smooth (that is, the partial derivatives always exist). If you have had a course in economics, you will recognize the importance of the marginal productivity of labor and its effect on wages. The Cobb-Douglas function can be applied to relatively simple models such as the observable result that the early introduction of word processors into a newsroom will raise productivity far greater than later upgrades. Also the Cobb-Douglas function can be used on an international scale. For instance the application of capital expenditures can have a far greater impact on rug-making in India where there is still a great deal of hand labor than in the United States where capital expenditure on machinery is already very large.

You will encounter the Cobb-Douglas model repeatedly in economics and business where the concept of constant returns to scale is required and the property of smoothness makes the model simple to use.