Lab 6: The Derivative

What does it mean for a line to be tangent to the graph of a function of one variable? Do the old notions of tangency extend to any function? Why are we interested? We want to know how the functions are changing! Is an investment increasing or decreasing? How fast is it changing? Is the target population for your product growing? Is a marketing strategy working?

The derivative is the slope of the tangent line to the function at a point in its domain. It gives the information about the rate of change at that point. But first we need to know what tangency means in this context.

1. What is the tangent line to the graph of a function?

To get a feel for what tangent lines to curves should look like, open the **Tangents** tool in the **Derivative Kit** and move the cursor over the upper graph. The straight line that moves along the curve is the tangent line. You might have had a chance to investigate this tool thoroughly in an earlier lab. For now, you should try a few examples to see if the tangent line shown on the upper graphs meets your expectations.

The purpose of this lab is to help you understand how the derivative of a function f is found at a given point x in the domain of f. The **derivative of *f* at** x is precisely the slope of the tangent line to the graph of f at $(x,f(x))$. Interestingly, first we find the derivative as a limit of the slopes of secant lines defined below and then we construct the tangent line using the derivative as the slope.

Now we will start working toward the definition of the derivative:

1.1 Open the **Definition of the Derivative** tool. Consider the points $(x,f(x))$ and $(x+h,f(x+h))$. On the upper graphing window, move the h slider to the right of x, and then move the h slider toward x, so that point $(x+h,f(x+h))$ slides along the curve toward point $(x,f(x))$. Describe what happens to the secant line from $(x,f(x))$ to $(x+h,f(x+h))$.

1.2 Now start with the h slider to the left of x, and then move the h slider toward x, so that point $(x+h,f(x+h))$ again slides along the curve to point $(x,f(x))$. Describe what happens to the secant line from $(x,f(x))$ to $(x+h,f(x+h))$.

1.3 What is the slope of the secant line from $(x, f(x))$ to $(x + h, f(x + h))$?

rise =_________

run =_________

$m_{\sec} = \frac{rise}{run} =$ _________________

The formal definition for the derivative of *f* at x, that is, the slope of the tangent line to the graph of f at $(x, f(x))$ is given below. Note that it's just the limit of the slopes of the secant lines from $(x, f(x))$ to $(x + h, f(x + h))$ as $(x + h, f(x + h))$ slides toward $(x, f(x))$ from either side.

The Definition of the Derivative of *f* at *x*

The derivative of a function f at a point x in its domain, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad (1)$$

provided that the limit exists.

1.4 The tangent line is the line through the point $(a, f(a))$ on the curve whose slope is $f'(a)$. Write the equation of that line.

1.5 Select the function $f(x) = x^3 + 3x^2 + 1$ from the function list on the tool. What is the slope of the secant line corresponding to $x = -1$ and $x + h = 1$?

Move $x + h$ toward x with the h-slider. What is the derivative at $x = -1$?

What is the equation of the line tangent to the function $y = x^3 + 3x^2 + 3$ at any point a in its domain?

2. When does a derivative fail to exist?

2.1 Try examples from the list to discover those situations when the limit does not exist. There are three situations in which a is in the domain of f but $f'(a)$ does not exist. For each situation, give a function in the list and the point in the domain for which the derivative does not exist. This exercise may take some experimentation on your part.

a. The function f is not continuous at a .

b. The function f has a vertical tangent at $(a, f(a))$, that is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty .$$

c. The function f has a **cusp** or a **corner** at $(a, f(a))$. (That is the right hand limit for (1) and the left hand limit for (1) are finite but unequal or both limits are infinite with different signs.)

We call a function f **differentiable** if it has a derivative $f'(a)$ at every point a in its domain. It can be shown that differentiable functions are always continuous but, as you can see, the possibility of vertical tangents, corners or cusps prevents the converse from always being true. That is, there are continuous functions that are not differentiable. We'll see that differentiability guarantees **smoothness** in that there are no sharp turns or vertical tangents.

3. The derivative function

As you experiment with the **Definition of the Derivative** tool, notice that a function is being traced out in the lower graph. For each point for which the limit in (1) exists, there is a corresponding point on the curve in the lower graph. We will call this new function the **derivative function** f' **.**

3.1 What happens on the lower graph as $x + h$ slides toward x ?

3.2 Experiment with the function 10 , 11 and 12. Determine the x-values for which the derivative function does not exist.

There are alternate visualizations of the derivative that may help you develop a feel for this material. We will explore one of the most inherently visual techniques.

4. The Zoom Method

Open the **Tangent Zoom** tool.

4.1 a.. Move the a -slider to 1 and hit the **zoom in** button to zoom in on the rectangle about the point. Keep zooming in on the original point. What happens to the curve passing through that point as you zoom in?

b. Try other points along the curve. What happens as you zoom in?

The zoom method allows us to visualize the fact that a function f has a derivative at a provided that it is "eventually linear" at $(a, f(a))$, that is, as you zoom in the function looks more and more like a non-vertical straight line. In fact it looks more and more like the line that is tangent to the function at this point. Then the derivative is just the slope of this line. If the function f is a differentiable function, then it is eventually linear at every point in its domain.

The only difference between the concepts of differentiability and essential linearity is in the case where the tangent line is vertical. The function is still essentially linear but the derivative fails to exist!

4.2 Now look at the lower screens:

Select the function $f(x) = x^3 + 3x^2 + 1$. Pick a point on the graph of the function and zoom in until the graph in the derivative zoom window is a horizontal line. How is this horizontal line related to the straight line in the window above it?

Now we will consider how we can detect points on the curve where the derivative fails to exist.

It's clear that if the function is discontinuous at the point, then every zoom image of the curve through that point will also be discontinuous there, so the curve will never approach a (continuous) straight line through the point.

Now we will look at the possible ways a continuous function f could fail to have a derivative at a given point a in its domain.

4.3 Select the function $f(x) = 1 - x^{2/3}$. Set $a = 0$ using the a -slider. Now zoom in on the point (0,1).

a. Describe what happens to the curve at that point as you zoom in? (Is it "eventually linear"?)

b. What behavior described in exercise 2.1 does this function exhibit at the point (0,1)?

4.4 Select the function $f(x) = abs(x)$, the absolute value of x.

Where do you expect the function to fail to have a derivative? Find that point and zoom in. Describe what happens.

5. Conclusion

We've experimented with two visualizations for the tangent to the graph of a function at a point. The first method is the basic one and leads to the standard definition. Because the concept of the derivative as the slope of the tangent line to the curve is so fundamental, it's important to try to visualize the process in a variety of ways. The fact that the graph of a differentiable function looks more and more like the tangent line to the curve at a point on its graph as we zoom in on the point allows us to use the tangent line to approximate the function close to the point. This idea will be a recurrent theme in later work.