Lab 20: Polynomial Approximations of Functions

In previous lab explorations, we demonstrated the characteristics of various functions and how they can be applied to model real data. A very serious question is to ask is: Are we able to integrate each of the functions we have seen as mathematical models?

Let us consider the function xe^{-x}, or sin x. Perhaps we would like to integrate this function but do not know a technique. One possibility is to use numerical integration, such as the Trapezoidal Rule. However, there are other methods of approximation that are very helpful.

There is some very good news regarding these questions!!! Here it is:

Any function for which the nth derivative can be found, can be approximated using a polynomial of at most the nth degree. The approximation will be valid in some interval in the domain of the function.

1. Approximating the sine function with a polynomial.

Let us begin by considering the function $y = \sin x$ in the interval $-0.5 \le x \le 0.5$ and an approximation to this function using the simple monomial (one-term polynomial) $y = x$.

1.1 Using your calculator, complete the table of values below. Note that x values are in radians.

x	$y =$ sin x	$y = x$
-0.5		
-0.25		
0.0		
0.25		
0.5		

1.2 From the table above, find the maximum error in using $y = x$ to approximate sin x in this interval.

Maximum error: ____________________

1.3 Now let us repeat this process for the same function using a cubic polynomial: $y = x - x^3/6$.

x	$y = \sin x$	$y = x - x^3/6$
$-\ 0.5$		
$-\ 0.25$		
0.0		
0.25		
0.5		

Maximum error: ____________________

1.4 Which approximation is better in the interval? __________________________

1.5 If we added more terms with higher powers to the polynomial, what do you think will happen?

The two polynomials we have used to approximate the sine curve near zero are first degree and third degree *Maclaurin Polynomials*. These are very specific polynomials, found using the derivatives of a function. In this lab, we will visually explore the fit of a polynomial to some other function. The nth degree ***Maclaurin Polynomial*** for a function $f(x)$ is a polynomial of the form

$$P_n(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \quad f^{(n)}(0)x^n/n!$$

Open the Series Kit and choose the **Maclaurin Tool**

The opening function is $f(x) = xe^{-x}$. When you choose a degree for the polynomial, the graph of the function will also appear.

2. Finding Maclaurin Polynomials

2.1 Click on $\boxed{\text{first}}$ to see the first degree polynomial and its graph. Write the first degree polynomial approximation.

$y =$ _____________________________

2.2 For what interval of x values is this linear function a reasonable approximation for the function?

2.3 Click on $\boxed{\text{third}}$ to see a cubic polynomial approximation. Give the equation of the 3^{rd} Maclaurin Polynomial:

$y =$ ______________________________

2.4 For what interval of x values is is this cubic function a reasonable approximation for the function?

2.5 In the lower window, the graph of the actual error is shown in the same color as the polynomial, $f(x) - p(x)$. This error is the difference between the functionn and the polynomial approximation of the function. We can estimate the error by choosing a subsequent term of the polynomial. Try values of n on the slider to find the term which best approximates the error, shown in white. What is the term which best approximates the error?

err(x): ____________________

2.6 Finally, find the 5^{th} Maclaurin polynomial approximation.

$y =$ ______________________________

2.7 Find the value of n for which the error approximation is the smallest and specify the error term.

err(x): ____________________

2.8 Find the interval of $x -$ values for which the 5th degree polynomial is a reasonable approximation for the function.

2.9 Choose another function from the list, and repeat problems 2.6 – 2.8 for this function.

2.10 What relationship do you notice among the polynomial, the degree, and the best estimate of the error of approximation?

The results of this exploration leads to a general result: The error of approximation is no worse than the value of the next term in the polynomial,

$$\text{error: } |R_n(x)| \leq M|\, x^{n+1}|/(n+1)!$$

where M is a real number such that $|f^{(n+1)}(t)| \leq M$ for all t between 0 and x. That is, the error can be estimated using a constant multiple of the next term of the polynomial. The smaller the value of M, the better the estimate. So we generally try to find the smallest value of M satisfying the inequality.

insert an example

3. Taylor Polynomials

The Maclaurin approximation is very good for values of x near zero. Notice that the formula for finding a Maclaurin polynomial involves evaluation of the function and its derivatives at $x = 0$. Suppose, however, that we want to find a polynomial approximation for some other x value, say $x = a$. In this case, we use a *Taylor Polynomial* to find the polynomial. The n^{th} **Taylor polynomial** of f at a is given by:

$$P_n(x) = f(a) + f'(a)(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \cdots + \tfrac{f^{(n)}(a)}{n!}(x-a)^n$$

Again, the error of approximation is no worse than the value of the next term in the polynomial

$$\text{error: } |R_n(x)| \leq M|\,(x-a)^{n+1}|/(n+1)!$$

where M is a number such that $|f^{(n+1)}(t)| \leq M$ for all t between a and x. This approximation is very good for values of x near a.

Open the **Taylor Polynomial** Tool.

As you roll the cursor over the upper plane, the n^{th} Taylor Polynomial is drawn, and the value of a is shown below. In the lower plane, the error term is drawn. For those values of x where the error is zero or negligible, the polynomial is a good fit. For values farther away from a, the polynomial does not continue to follow the curve.

Fitting a Taylor Polynomial to a function

3.1 Using the function $f(x) =$ sin x, and a third degree polynomial, for what interval near $a = \pi$ is the polynomial a very good fit for the function?

x-values in the interval ________________________________

3.2 Use the interval in 3.1 to find the largest value of $|f^{(n+1)}(t)|$ in that interval.

$|f^{(n+1)}(t)| \leq$ ___________________________

3.3 Use this value for M in the formula for the error, and calculate the maximum error using this third degree Taylor Polynomial. (Choose a value of x in the interval which gives the maximum value of $f^{(n+1)}(x)$

Maximum error: _________________________________

3.4. Now, using the slider on n, choose $n = 6$ to find the sixth degree Taylor Polynomial.

$P_6(x) =$ ____

3.5 Repeat problems 3.1 through 3.4 to find the interval of fit and the maximum error in this interval.

interval of x-values ________________________________

$|f^{(n+1)}(t)| \leq$ ___________________________

Maximum error: _________________________________

3.6 Does the interval for which the polynomial is a good depend upon the choice of a? (Try another value a.)

3.7 Choose another function from the list and repeat problem 3.5.