Differential Geometry and its Applications. By John Oprea. The Mathematical Association of America, 2nd edition, 2007, xxi+469 pages, ISBN 978-0883857489, $69.95.

Reviewed by: Kristopher Tapp

What is the best course to help students make the transition from multivariable calculus and linear algebra (and possibly algebra or analysis) into higher abstract mathematics? What should we require of such a course? Here is my wish list. It should help free the students from compartmentalization by drawing on many past courses and by frequently combining tricks and techniques from multiple prerequisite topics into a single proof. The content should be directly relevant to as many areas of graduate-level mathematics as possible, and hopefully also to some areas of science. It should be grounded with real-world applications. There should be elegant proofs. There should be computations. It should be geometric. It should be algebraic. It should tell a compelling tale, whose plot line is not littered with too many unjustifiable black boxes. It should exemplify the rigor of higher mathematics, with rigor and intuition reinforcing each other throughout.

The subject of differential geometry is an attractive choice. This subject interweaves ideas of calculus, differential equations, and linear algebra into a beautiful story. This story begins with some characters familiar from multivariable calculus: parametric curves and surfaces. Then, using a bit of linear algebra and differential equations, we are introduced to curvature, geodesics, and parallel transport. These basic concepts are visually intuitive, with each intuition supported by a simple precise definition. The plot quickly progresses to some of the most striking theorems in global geometry, including the Gauss-Bonnet theorem.

Differential geometry has strong historical connections to physics, but many textbooks minimize these links, aiming instead for an uncluttered self-contained mathematical text. John Oprea's textbook is different. Applications to science abound. This changes the essential flavor of the topic. The reader comes to see differential geometry as, among other things, a language for solving practical problems that arise in nature.

For example, only nine pages into the text, Oprea derives the equation of the tautochrone. This is a U-shaped plane curve contrived so that the time needed for gravity to pull a frictionless object along the curve to its vertex does not depend on how high up the object starts. In the 1670s Christian Huygens attempted to use a tautochrone (together with involutes, which are also useful in constructing tooth shapes for interlocking gears) to design a clock which would function onboard a ship, thus allowing early sailors to determine their longitude at sea. In the next several pages, Oprea also derives the shape of a suspension bridge and the form of a heat-seeking missile's pursuit curve. I intend to use (or at least mention) some of these applications to enrich the curves section of my next multivariable calculus course.

At several points, I was pleasantly surprised to learn applications of topics which I'd considered to be pure mathematics. For example, after deriving the Delaunay surfaces from a pure math vantage point, by seeking surfaces of revolution with constant mean curvature, Oprea mentions their relationship to the roulette of an ellipse, and then provides a reference about "how surface tension and pressure combine to determine the shapes of various one-celled creatures – shapes suspiciously Delaunayan!"

Another pretty example involves the parallel extension of a vector around a latitude circle on the sphere. This means that the vector is extended along the latitude such that it is always tangent to the sphere, and such that its derivative is always normal to the sphere. Along the equator, the parallel extension of a north-pointing vector remains north-pointing, but along other latitudes it does not. For example, at 48 degrees (which on the earth is the latitude of Paris), the ending vector is turned a full 267 degrees from the starting vector. Although students will find this question geometrically natural, Oprea gives them another reason to care about its solution: Foucault's pendulum! In the 1850s Jean Foucault demonstrated the rotation of the earth by using a pendulum made from a heavy iron ball attached by a long wire to the dome of the Pantheon in Paris. Because of the earth's rotation, as time passes, the swing plane of such a pendulum rotates at a speed depending on the latitude where the pendulum sits. In Paris, this rotation is about 11.1 degrees per hour, or 267 degrees per day, the same number as above because the changing swing plane as the earth rotates equals the parallel extension of the initial swing plane around the latitude of the globe.

Oprea's text provides many more applications: the best shape for a nuclear cooling tower, the design of an industrial plastic-wrapping machine, the shape of a mylar balloon, and the famous brachistochrone problem. He includes a chapter on the calculus of variations, which further allows him to showcase differential geometry as a tool for solving practical physics and engineering problems.

These applications certainly strengthen and enrich the text. Many of them require explicit solving of differential equations. Oprea is not shy about explicit calculations! He includes a section on elliptic functions and scatters throughout the text many algebraic tricks for equation solving, by sweat and by Maple. In fact, the book is prefaced with the following advice to students: "Look for the right differential equations and then try to solve them, analytically or numerically, to discover the underlying geometry." This advice initially surprised me. I prefer my differential equations to hide in the background, guaranteeing the existence of geodesics and such, so that I'm free to focus on geometric proofs. But for an undergraduate text on this topic, Oprea's emphasis on explicit differential equations and local coordinate computations is probably appropriate.

The book contains other goodies, such as an appendix of student projects, and color images of minimal surfaces. Moreover, each chapter ends with a section in which Maple is used to compute and visualize certain topics from the chapter. Some of these Maple programs do a splendid job of grounding the abstract concepts, and helping students visualize things like the Clairaut relation. Even if you choose to teach out of a different text, buy yourself a copy of Oprea's book – it's a treasure trove of interesting, well-thought-out supplementary material.

I must end with mixed reviews as to the book's clarity. The author's style is conversational, which is often good. Oprea does an admirable job of providing plain English motivations and explanations of the mathematical concepts. He helps students along with useful explanations like "…curvature measures the deviation of a curve from being a line and torsion the deviation of a curve from being contained in a plane." I love it.

But at other points, the conversational explanations replace rigor rather than supporting it. Many important definitions, which deserve to be singled out and numbered, are instead blended into the running text. Key vocabulary terms are often defined (or at least first used) in the middle of a discussion that is more difficult to understand than the term itself. For example, a student first encounters the all-important term *covariant derivative* applied to an overly particular example of a covariant derivative. She is left wondering: what may one take the covariant derivative of? Two hundred pages later, the topic is revisited, and she finds this: one can take the covariant derivatives of [not necessarily tangent] vector fields on surfaces. Good, but the next exercise asks her to prove that a geodesic's velocity field has zero covariant derivative, so she guesses that one can also take the covariant derivative of a *vector field along a curve*. In fact, doing so is easy, so she worries that she may have cheated by ignoring that confusing technical bit about extending the vector fields to open sets in R^3. She moves on to *parallel transport* and *holonomy,* which are again defined in the running text. This time the running text contains local coordinate expressions, so she's unsure whether these fundamental notions depend on the choice of local coordinates. And must the local coordinates have that "F=0" property? She decides that holonomy must depend on the local coordinates, since it is defined along arbitrary paths (not just loops). But her decision seems incompatible with the very next theorem: isometries preserve holonomy. So she backtracks to the definition of *isometry*, and so on. I worry that strong students could sink under this added weight of carrying along their own evolving personal definitions of key vocabulary terms. Weak students might not even understand that something is missing.

In general the book is well written and well organized and treats many interesting topics. I would seriously consider teaching from Oprea's text because of its many strengths. My above concerns about clarity could be mitigated through lectures.

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