NONNEGATIVELY CURVED VECTOR BUNDLES WITH LARGE NORMAL HOLONOMY GROUPS

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Abstract. When B is a biquotient, we show that there exist vector bundles over B with metrics of nonnegative curvature whose normal holonomy groups have arbitrarily large dimension.

1. Introduction

Let M be an open manifold of nonnegative curvature with soul $\Sigma \subset M$. The *normal holonomy group*, Φ, of M is the group of all endomorphisms of a fixed fiber of the normal bundle of Σ in M induced by parallel transport around piecewise smooth loops in Σ. Its study is important because the large scale geometry of M is restricted by Φ. For example, if Φ is trivial, then M splits locally isometrically over its soul [3],[11]. If Φ is large, then the volume growth of M is small [5].

There is no restriction on the groups which occur. Every path-connected subgroup of $SO(k)$ occurs as the normal holonomy group of some simply connected manifold with nonnegative curvature. Wilking demonstrated this in [7] by showing that for a compact Lie group G, proper closed subgroup $H \subset G$, and isometric effective action of H on $(\mathbb{R}^k, \text{flat})$, the normal holonomy group of the orbit space

$$M = ((G, \text{bi-invariant}) \times (\mathbb{R}^k, \text{flat}))/H$$

is isomorphic to H, provided that the Lie algebra of H does not contain an ideal of the Lie algebra of G.

However, this result gives very limited information about which normal holonomy groups occur over a fixed soul. Among bundles over S^2, it says only that $SO(2)$ occurs. On the other hand, it is know that $S^2 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^4$ admit nonnegatively curved metrics with transitive normal holonomy groups [6],[9]. The following proposition improves the situation somewhat.

Proposition 1.1. *Let $B = G//H$ be a compact biquotient. Then there exist vector bundles over B with metrics of nonnegative curvature whose normal holonomy groups have arbitrarily large dimension, provided the vertical space at e of $G \to G//H$ does not contain an ideal of the Lie algebra of G.*

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To prove Proposition 1.1, we first generalize Wilking's above-mentioned result on homogeneous vector bundles to biquotient vector bundles:

Proposition 1.2. *Let $B = G//H$ be a compact biquotient. Let H act isometrically and effectively on $(\mathbb{R}^k, \textit{flat})$. Then the normal holonomy group of the orbit space*

$$M = ((G, \textit{bi-invariant}) \times (\mathbb{R}^k, \textit{flat}))/H$$

is isomorphic to H, provided the vertical space at e of $G \to G//H$ does not contain an ideal of the Lie algebra of G.

Aside from the examples of [1] and [10], biquotient bundles as in Proposition 1.2 are the only vector bundles know to admit nonnegative curvature, which justifies studying their geometry. Proposition 1.1 is a consequence of Proposition 1.2 because any biquotient can be given increasingly complicated biquotient descriptions, as Wilking showed in [9]:

$$G//H = \Delta(G)\backslash(G \times G)/H = \Delta(G \times G)\backslash(G \times G \times G \times G)/\Delta(G) \times H = \cdots$$

Unfortunately, Proposition 1.2 does not help decide whether transitive normal holonomy groups (or large dimensional *simple* normal holonomy groups) occur in high-rank vector bundles over a fixed soul. These problems are open for S^2. We prove the following negative result for connection metrics, the case $k = 3$ of which appears in [6]:

Proposition 1.3. *For a connection metric with nonnegative curvature on an $\mathbb{R}^k$ bundle over S^2, every irreducible subspace for the action of the normal holonomy group has dimension 1 or 2.*

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2. Background on Holonomy Groups

Suppose that $\pi : P \to \Sigma$ is a principle bundle with structure group H (acting freely on P on the left). Assume that P has a nonnegatively curved H-invariant Riemannian metric, and that Σ has the induced metric for which π is a Riemannian submersion. The horizontal distribution of π induces a principle connection on P. Any piecewise smooth loop α at a point $b \in \Sigma$ induces a diffeomorphism, f_α, of the fiber $F_b := \pi^{-1}(b)$ via horizontal lifts. The group of all such diffeomorphisms is called the *holonomy group*, $\tilde{H}$, of π (in general, this strategy defines the holonomy group of a Riemannian submersion or of a principle connection; in this case, they are the same).

We can regard $\tilde{H}$ as a subgroup of H; that is, the holonomy group of a principle connection is a subgroup of the structure group. This standard fact is justified as follows. Fist identify $H \cong F_b$ by identifying each $h \in H$ with $h \star p_0$, where $p_0 \in F_b$ is arbitrary and fixed, so $e \cong p_0$. If $\alpha : [0, 1] \to \Sigma$ is a loop at b, and $\overline{\alpha}$ is a horizontal lift of α beginning at $\overline{\alpha}(0) = e$, then the holonomy element $f_\alpha \in \tilde{H}$ acts on F_b as right-multiplication by $\overline{\alpha}(1)$. This is because for any $g \in H$, $g \star \overline{\alpha}$ is a horizontal lift of α beginning at $g \in F_b$, so $f_\alpha(g) = g\overline{\alpha}(1)$. In summary, we identify the holonomy group, $\tilde{H}$, with the group of (right multiplications by) endpoints in F_b of piecewise smooth horizontal paths beginning at $e \in F_b$.

Given an orthogonal effective action of H on $\mathbb{R}^k$, we can form the associated bundle:

$$M = P \times_H \mathbb{R}^k = (P \times \mathbb{R}^k)/H.$$

The induced Riemannian submersion metric on M (using the flat metric on $\mathbb{R}^k$) is nonnegatively curved. The zero-section, $\Sigma \cong P \times_H \{0\}$, is a soul of M, where $0 \in \mathbb{R}^k$ is any fixed point of the H-action. The Sharafutinov retraction onto the soul equals the bundle projection map $\mathrm{sh} : M \to \Sigma$, which has the simple description $\mathrm{sh}([p, v]) = [p, 0]$. Here we use square brackets to denote equivalence classes under the H-action.

So M can be regarded as a vector bundle over Σ (with a connection inherited from the principle connection), or as a manifold with nonnegative curvature whose soul is Σ. The connection in the normal bundle of the soul is the same as this connection inherited from the principle connection. Its holonomy group again just equals $\tilde{H}$:

Lemma 2.1. *The holonomy group of* $sh : M \to \Sigma$ *is isomorphic to* $\tilde{H}$.

Proof. Consider the Riemannian submersions

$$P \times (\mathbb{R}^k, \mathrm{flat}) \xrightarrow{\Pi} M \xrightarrow{\mathrm{sh}} \Sigma.$$

Remember that π denotes the projection $\pi : P \to \Sigma$. As before, choose $b \in \Sigma$ and identify $\tilde{H} \subset H \cong F_b = \pi^{-1}(b)$. Define $\Psi : \mathbb{R}^k \to \mathrm{sh}^{-1}(b)$ as the natural diffeomorphism sending $v \mapsto [e, v]$.

Any loop $\gamma : [0, 1] \to \Sigma$ at b has the form $\gamma(t) = [\alpha(t), 0]$, where $\alpha(t)$ is a π-horizontal path in P with $\alpha(0) = e$ and $\alpha(1) \in \tilde{H}$. For fixed $v \in \mathbb{R}^k$, the path $\overline{\gamma}(t) = [\alpha(t), v]$ is a sh-lift of γ to M with $\overline{\gamma}(0) = \Psi(v)$. Further, $\overline{\gamma}$ is sh-horizontal because $t \mapsto (\alpha(t), v)$ is a Π-lift of $\overline{\gamma}$ which is everywhere orthogonal to the $(\mathrm{sh} \circ \Pi)$-fibers.

Notice that $\overline{\gamma}(1) = [\alpha(1), v] = [e, \alpha(1)^{-1}(v)] = \Psi(\alpha(1)^{-1}(v))$. It follows that the holonomy group of sh is isomorphic to $\tilde{H}$ acting on a fiber as the inverse of the given action. ☐

The above lemma says that a principle connection has the same holonomy group as the connection it induces in an associated bundle. The nonnegative curvature hypothesis on P was not relevant for this story, but is crucial in the next lemma, which is a special case of Wilking's main theorem in [9]. It says that any horizontal vector paired with any vertical vector orthogonal to the holonomy orbit must generate a flat. As before, we identify $\tilde{H} \subset H \cong F_b$.

Lemma 2.2 (Wilking)**.** *Let* $X, V \in T_eP$ *with* X *orthogonal to* H *and* V *tangent to* H *but orthogonal to* $\tilde{H}$*. Then* $span\{X, V\}$ *exponentiates to a totally geodesic flat in* P*.*

3. HOLONOMY IN BIQUOTIENT BUNDLES

In this section, the principle bundles we consider will be biquotient bundles. For such bundles, we show under very general conditions that the holonomy group acts transitively on a fiber; that is, $\tilde{H} = H$. We use this to prove Propositions 1.1 and 1.2.

Let G denote a compact Lie group with Lie algebra $\mathfrak{g}$. Let $H \subset G \times G$ be a closed subgroup. Assume that the following action of H on G is free: $(h_1, h_2)\star g := h_1 \cdot g \cdot h_2^{-1}$. The orbit space is called a *biquotient*, and is denoted $G//H$. The projection map $\pi : G \to G//H$ is a principle H-bundle.

Let g_0 be a bi-invariant metric on G, so the projection

$$\pi : (G, g_0) \to (G, g_0)//H$$

is a Riemannian submersion. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal distributions of π. Let $\tilde{H} \subset H$ denote the holonomy group; that is, the set of elements $h \in H$ such that $h \star e = \alpha(1)$ for some piecewise smooth π-horizontal path α in G beginning at $\alpha(0) = e$. The following lemma provides a sufficient condition for the holonomy group to act transitively.

Lemma 3.1. *If $\mathcal{V}_e$ does not contain an ideal of $\mathfrak{g}$, then $\tilde{H} = H$.*

Proof. Let $\mathfrak{m} \subset \mathcal{V}_e$ denote the space of vertical vectors orthogonal to the holonomy orbit $\tilde{H} \star e$. It will suffice to prove that $\mathfrak{m} = \{0\}$.

Suppose there exists a non-zero vector $V \in \mathfrak{m}$. By Lemma 2.2, for all $X \in \mathcal{H}_e$, $\sigma = \text{span}\{X, V\}$ is a zero-curvature plane, so $[X, V] = 0$. Let $\alpha(t) = \exp(tX)$ and let $V(t)$ be the parallel field along $\alpha(t)$ with $V(0) = V$. Since σ is tangent to a totally geodesic flat in (G, g_0), $V(t)$ is both right and left invariant along $\alpha(t)$. In particular, $\text{Ad}_{\alpha(t)}V = V$ for all t.

More generally, if $\alpha(t)$ is any horizontal piecewise geodesic path in G, then $\text{Ad}_{\alpha(t)}V = V$ for all t. To see this, suppose that $\{\alpha(t_1), \alpha(t_2), ...\}$ are the non-smooth points of α. Let $V(t)$ be the parallel transport of V along $\alpha(t)$. As shown in [8], $V(t)$ remains orthogonal to the holonomy orbits, so $\sec\{Y, V(t_1)\} = 0$ for all horizontal vectors Y, particularly for the right-derivative $Y = \alpha'(t_1^+)$. Then $V|_{[t_1,t_2]}$ is left (respectively right) invariant along $\alpha|_{[t_1,t_2]}$ because left (respectively right) multiplication by $\alpha(t_1)^{-1}$ sends it to a parallel field along a geodesic which exponentiates to a flat, and is therefore left (respectively right) invariant.

Therefore, $\text{Ad}_g V = V$ for all $g \in \tilde{H} \star e$. It follows that $[Z, V] = 0$ for all $Z \in \mathcal{V}_e$ tangent to $\tilde{H} \star e$. Thus, $\mathfrak{g} = \mathfrak{m} \oplus (\mathfrak{m}^\perp)$ is a decomposition of $\mathfrak{g}$ into orthogonal commuting subspaces. Using the bi-invariance of the metric, it follows that $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are both ideals of $\mathfrak{g}$, which contradicts the hypothesis, since $\mathfrak{m} \subset \mathcal{V}_e$. □

In Lemma 3.1, the identity element $e \in G$ is not distinguished. If for some $g \in G$, $\mathcal{V}_g$ (pulled back to e via left or right multiplication) contains no ideal of $\mathfrak{g}$, then we can similarly conclude that $\tilde{H} = H$.

Proposition 1.2 follows immediately from Lemmas 2.1 and 3.1. To obtain Proposition 1.1 from Proposition 1.2 as explained in the introduction, it remains only to verify that:

Lemma 3.2. *If the vertical space at e of $G \to G//H$ does not contain an ideal of $\mathfrak{g}$, then the vertical space at (e,e) of $G \times G \to \Delta(G)\backslash(G\times G)/H$ does not contain an ideal of $\mathfrak{g}\oplus\mathfrak{g}$.*

Proof. The diffeomorphism from [9] used to prove that

$$\Delta(G)\backslash(G\times G)/H \approx G//H$$

is induced by that map $f : G\times G \to G$ defined as $f(g_1,g_2) := g_1^{-1}\cdot g_2$. Notice that $df(X,Y) = Y - X$ for all $(X,Y) \in \mathfrak{g}\oplus\mathfrak{g}$. By construction, df maps the vertical space at (e,e) of $G\times G \to \Delta(G)\backslash(G\times G)/H$ to the vertical space at e of $G \to G//H$. Further, df sends any ideal of $\mathfrak{g}\oplus\mathfrak{g}$ to an ideal of $\mathfrak{g}$. The result follows. □

4. Connection metrics in bundles over S^2

In this section, we prove Proposition 1.3.

Proof. Let M denote the total space of a vector bundle over S^2 with a connection metric of nonnegative curvature. Let $\Sigma \subset M$ denote a soul of M. Let R^∇ denote the curvature tensor in the normal bundle, $\nu(\Sigma)$, of Σ. For $p \in \Sigma$, the skew-symmetric map $R^\nabla_p(X,Y) : \nu_p(\Sigma) \to \nu_p(\Sigma)$ does not depend on the choice of oriented orthonormal vectors $X, Y \in T_p\Sigma$, so we denote this map simply as $R^\nabla_p : \nu_p(\Sigma) \to \nu_p(\Sigma)$.

The main inequality of [4] implies that for all $p \in \Sigma$, $V_1, V_2 \in \nu_p(\Sigma)$, and unit-length $X \in T_p\Sigma$,

$$\langle (D_X R^\nabla)(V_1), V_2\rangle^2 \le K_p \cdot \langle R^\nabla(V_1), V_2\rangle^2,$$

where K_p denotes the sectional curvature of Σ at p.

Now fix $p \in \Sigma$. Choose $V_1, V_2 \in \nu_p(\Sigma)$ and let $V_1(t), V_2(t)$ denote their parallel transports along a piecewise-geodesic loop $\alpha(t)$ in Σ at p. Define $f(t) = \langle R^\nabla(V_1(t)), V_2(t)\rangle$. The above inequality implies that

$$f'(t)^2 \le K_{\alpha(t)} \cdot f(t)^2.$$

If $f(0) = 0$, then this differential inequality implies that $f(t) = 0$ for all t. So if $\langle R^\nabla_p(V_1), V_2\rangle = 0$ and $A : \nu_p(\Sigma) \to \nu_p(\Sigma)$ denotes the parallel transport along a loop in Σ, then $\langle R^\nabla_p(A(V_1)), A(V_2)\rangle = 0$ as well.

Let W_0 be the nullspace of R^∇_p. Since R^∇_p is skew-symmetric, there exists an orthogonal splitting of $\nu_p(\Sigma)$ into invariant subspaces for R^∇_p of the form $\nu_p(\Sigma) = W_0 \oplus W_1 \oplus \cdots \oplus W_l$, where $\dim(W_i) = 2$ for $i > 0$.

The nullspace W_0 is invariant under the holonomy group, Φ, of $\nu(\Sigma)$. It determines a sub-bundle of $\nu(\Sigma)$ whose induced connection is flat, so Φ acts trivially on W_0. It remains to determine how Φ acts on $W := W_1 \oplus \cdots \oplus W_l$.

If $V_1 \in W_i$ and $V_2 \in W_j$ with $i \neq j$, then $\langle R_p^\nabla(V_1), V_2\rangle = 0$, and therefore $\langle R_p^\nabla(A(V_1)), A(V_2)\rangle = 0$ for all $A \in \Phi$. So for every $i > 0$ and for every $A \in \Phi$, $R_p^\nabla(A(W_i)) = A(W_i)$. That is, A sends each W_i to a R_p^∇-invariant subspace.

With resect to an orthonormal basis of vectors from the subspaces W_i,

$$R_p^\nabla|_W = \operatorname{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & a_l \\ -a_l & 0 \end{pmatrix}\right).$$

If all a_i's are distinct, then the planes W_i are the only 2-dimensional R_p^∇-invariant subspaces of W, so $A(W_i) = W_i$ for every $i > 0$ and every $A \in \Phi$, which completes the proof.

Suppose that not all a_i's are distinct; for example, $a_1 = \cdots = a_{l'}$, and this value is distinct from the other a_i's. Denote $W' := W_1 \oplus \cdots \oplus W_{l'}$. Let $A \in \Phi$. The fact that $R_p^\nabla(A(W_i)) = A(W_i)$ for all $i > 0$ implies only that $A(W') = W'$ and that A is complex-linear on W'. So W' determines a sub-bundle of $\nu(\Sigma)$ with a parallel almost-complex structure such that $R_p^\nabla(V) = a_1 \cdot i \cdot V$ for all $V \in W'$.

If at all other points $q \in \Sigma$, R_q^∇ has only one (complex) eigenvalue on this sub-bundle, so that $R_q^\nabla(V)$ is a real multiple of $i \cdot V$ for all V in this sub-bundle, then the Ambrose-Singer Theorem implies that the holonomy group of this sub-bundle is one-dimensional, so $A(W_i) = W_i$ for all $i \in \{1, l'\}$. On the other hand, if R_q^∇ has distinct eigenvalues for some $q \in \Sigma$, then the above arguments split the sub-bundle into smaller sub-bundles with parallel almost-complex structures. This reduction gives in the end that $A(W_i) = W_i$ for all i.

$\square$

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