Math 2361
Logic and Foundations
Spring 2009

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MTWΘF	8:45–9:45
Mon.& Wed.	1:00–2:30
Friday	1:30–3:00

· · ·and by appointment or chance.

Text:
A. G. Hamilton.
Logic for Mathematicians.
(revised edition)
Cambridge: Cambridge University Press, 1988.

I. Introduction. At the end of the Nineteenth Century, mathematicians believed that they had resolved for all time questions about the logical foundations of mathematics: work of Cauchy, Weierstrass, Dedekind, and Cantor among others was demonstrating how to reduce all mathematics to set theory. Even the laws of logical thinking themselves had been clearly formulated (by Boole and Frege).

This eternal resolution lasted about a decade. Mathematics had suddenly become more general, powerful and abstract, and many mathematicians and philosophers doubted that the new mathematics was logically sound or even made sense. The discovery of Russell's Paradox fueled the unease by demonstrating that there was a flaw in the definition of a set; the definition was too broad. Mathematicians dealt with this difficulty by replacing the definition of "set" with list of axioms that are meant to capture enough of the old ("naïve") set theory to build the edifice of mathematics but also leave out enough to avoid contradictions[1]. Since there was no iron-clad guarantee that either of these objectives had been accomplished, doubts gained credibility and doubters multiplied. Turmoil over the logical underpinnings mathematics raged–that's not too strong a word–through the 1920s.

The main school attacking the established methods, "Intuitionism," held (among other things) that a certain accepted rule of logic, the "Law of the Excluded Middle," while obvious when applied to finite sets, was suspect when applied to infinite sets. Led by L.E.J. Brouwer, the Intuitionists set about trying to reconstruct mathematics using only restricted logic (without the suspect rule), but these attempts met with only very limited success: compared to standard (classical) mathematics, Intuitionist math was much harder, less elegant, and much weaker. (For example: to date, the Intuitonists have not been able to prove the Mean Value Theorem.)

Another school, the Formalists, saw a possible way to answer the Intuitionists. Led by David Hilbert, the Formalists attempted to resolve doubts about both logic and set theory by studying axiom systems themselves. Euclid was the first to organize a body of knowledge axiomatically, and nineteenth-century mathematicians had modernized his approach and applied it to many areas of mathematics. It was Hilbert's insight that yet another refinement of the axiomatic method had the potential to place all of mathematics–set theory, number theory, even mathematical logic itself–onto a rock-solid foundation which would put paid to the Intuitionists' disturbing questions. Roughly speaking, the new axiomatics of the formalists can be thought of as a sort of a robot, programmed with logic, that can automatically prove theorems within an axiom system. The robot was programmed with the more powerful logic of ordinary mathematics ("classical logic"); the goal was to use the limited Intuitionistic logic to prove that, using its more powerful classical

[1] Similarly, mathematicians avoid the difficulty of defining *probability* by axomatizing it instead.

logic, the robot: (a), was powerful enough to prove any true mathematical statement and to disprove any false one; and (b), was consistent—that is, was incapable of proving two contradictory statements.

After several decades of progress, the Formalists' hopes were dashed in 1931 by two theorems of Kurt Gödel, which show that the Formalist program is essentially hopeless. Roughly, Gödel's first theorem proves that no formal logical system of the sort constructed by the Formalists can do both (a) and (b) above; and and his second theorem shows: if it is the case that the robot is consistent, then there is no hope of proving this fact using Intuitionistic logic (or even such classical logic as had been programmed into the robot).

This course will develop the mathematics behind this story. We will cover the following topics:

- [**I**]**:** A rapid review of pre-axiomatic set theory;
- [**II**]**:** The paradoxes and the intuitionists' objections to classical mathematics;
- [**III**]**:** The Formalists' response: formal axiomatics and metamathematics
  - [**a**]: Propositions and the Propositional Calculus
  - [**b**]: Predicates and the Predicate Calculus
  - [**c**]: Some axiomatic set theory
  - [**d**]: Some axiomatic number theory
- [**IV**]**:** Gödel's theorems of 1931.

II. Mechanics.

Objectives. The successful student will understand the issues that led to the creation of the Formalists' methods; will understand how these methods work and be able to work with them; and will understand some of the properties and limitations of these systems, including Gödel's theorems of 1931.

Exams. There will be two take-home **midterm exams** and a take-home **final**. The midterms will each determine 20% of your grade; your final and your homework average will each determine 30% of it.

Homework. The homework is really the heart of the course. The only way to learn material of this difficulty is by doing; you must put in a lot of time concentrating on the concepts and techniques that will be flying at you in order to master them. This includes time spent staring at hard problems and maybe getting nowhere; this time is **NOT** wasted. As you struggle with problems, you are building mathematical muscles, and you are developing an intuitive understanding of what the definitions and the theorems *really* mean. These intuitions are essential to real knowledge and understanding of mathematics: they help you grasp and remember what you've already seen, and they lead you to guess the shape of mathematics yet to come.

The possible grades on any problem will be 0/4/7/9/10, where

"10" means "completely correct,"
"9" means "almost completely correct,"
"7" means "you made progress,"
"4" means "you got started," and
"0" means "you didn't get started."

Turn in your homework on $8\frac{1}{2}'' \times 11''$ paper, leaving the right third of each sheet blank; this leaves room for my comments. Also, preface each solution you hand in with a statement of the problem you are solving (including page number and problem number, when the problem comes from the text).

Resubmission of problems. You may resubmit any problem; your grade on the resubmission will replace your original grade on the problem.