Arclength

I. Introduction. To the eye, the graph of a "nice" function $y = f(t)$ defined on a closed interval $[a, b]$ appears to have a length. One way of measuring the length would be to lay a string out along the curve, to pull it taught, and to place it next to a ruler. Another would be to set the odometer of a car to zero, have the car drive along a road built in the shape of the curve, and record the reading of the odometer afterwards. Focus on this second way of measuring the length: this finds the length as the result of a *continuous accumulation*, over time, to the odometer. This means that length *ought* to be expressible as in integral of some well-behaved function $y = g(t)$, integrated from $t = a$ to $t = b$:

$$\text{length} = \int_a^b g(t)\,dt. \tag{1}$$

The purpose of this handout is to develop such an integral; but the attempt to do this runs into a serious problem, as you will see below. Moreover, in this much generality, the problem we encounter has no solution: if all we know about f is that it is continuous, length of the curve $y = f(t)$ will not necessarily be given by an integral like (1).* In order to get around the difficulty and push the development through, we will restrict our attention to a function f that is more than just continuous. We will need f to be not only continuous but also differentiable on $[a, b]$; furthermore, we will need to stipulate that the derivative f' of f be continuous there as well.

II. The problem. To detect the problem, let us start to construct an integral for length. We start much as we did to find integrals for area, volume, and work. Recall that the technique for finding an integral for a given quantity consists of the following steps: **(a)**, you partition the interval $[a, b]$ into n equal subintervals of length $\Delta t = \dfrac{b-a}{n}$; **(b)**, you use the partition to write down an approximation of the quantity being treated as a Riemann sum,

$$\text{quantity} \approx \text{approximation} = \sum_{i=1}^{n} g(t_i^*)\Delta t, \tag{2}$$

where g is a function defined on $[a, b]$ and t_i^* is a "sample point" in the i^{th} subinterval of the partition; and **(c)**, you take $\lim\limits_{n\to\infty}$, which is guaranteed to exist if g is continuous:

$$\text{quantity} = \lim_{n\to\infty}(\text{approximation}) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} g(t_i^*)\Delta t\right) = \int_a^b g(t)\,dt.$$

In the case at hand, the quantity we're treating is the length of the graph of $y = f(t), a \le t \le b$. First, we partition the interval:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

This gives us $n + 1$ points on the graph of f, namely

$$P_0 = \big(t_0, f(t_0)\big);\ P_1 = \big(t_1, f(t_1)\big);\ P_2 = \big(t_2, f(t_2)\big);\ \cdots\ P_{n-1} = \big(t_{n-1}, f(t_{n-1})\big);\ P_n = \big(t_n, f(t_n)\big).$$

Next, from the partition we construct an approximation of the length of the curve. The natural thing to take is the sum of the lengths of the n segments $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\cdots$, $\overline{P_{n-1}P_n}$:

$$\text{length} \approx \text{approximation} = \sum_{i=1}^{n} \overline{P_{i-1}P_i}.$$

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[*] In fact, there are even examples of continuous functions, defined on closed intervals $a \le t \le b$, the lengths of whose graphs are infinite!

Now, by the Pythagorean Theorem, it is easy to see that $\overline{P_{i-1}P_i} = \sqrt{\big(f(t_i) - f(t_{i-1})\big)^2 + \big(\Delta t\big)^2}$, so that

$$\begin{aligned} \text{length} \approx \text{approximation} &= \sum_{i=1}^{n} \overline{P_{i-1}P_i} \\ &= \sum_{i=1}^{n} \sqrt{\big(f(t_i) - f(t_{i-1})\big)^2 + \big(\Delta t\big)^2}. \end{aligned} \tag{3}$$

The problem is now visible: the approximation (3) is **NOT** of the required form (2). The summands in formula (3) have no factor "Δt" and nothing that looks like a candidate for the function g; hence, the theory of the integral cannot help us take the limit of (3) in its current form.

III. Fixing the problem. To get around this obstacle, we will need to use our additional assumptions as well as the Mean Value Theorem ("*MVT*"). Fix one subinterval $[t_{i-1}, t_i]$. Since function $y = f(t)$ is differentiable (hence continuous) on this interval, the MVT guarantees at least one point t_i^* in the interval for which

$$f'(t_i^*) = \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} = \frac{f(t_i) - f(t_{i-1})}{\Delta t},$$

so that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*) \cdot \Delta t. \tag{α}$$

We now can make the substitutions (α) into the last expression of equation (3):

$$\begin{aligned} \text{length} \approx \text{approximation} &= \sum_{i=1}^{n} \overline{P_{i-1}P_i} \\ &= \sum_{i=1}^{n} \sqrt{\big(f(t_i) - f(t_{i-1})\big)^2 + \big(\Delta t\big)^2} \\ \text{make substitutions} \longrightarrow &= \sum_{i=1}^{n} \sqrt{\big(f'(t_i^*) \cdot \Delta t\big)^2 + \big(\Delta t\big)^2} \\ \text{factor out } \big(\Delta t\big)^2 \longrightarrow &= \sum_{i=1}^{n} \sqrt{\big[(f'(t_i^*))^2 + 1\big]\big(\Delta t\big)^2} \\ \text{algebra} \longrightarrow &= \sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + 1} \cdot \sqrt{\big(\Delta t\big)^2} \\ \text{algebra} \longrightarrow &= \sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + 1} \cdot \Delta t. \end{aligned} \tag{4}$$

The last line of (4) above is what we need: this is of the form $\sum_{i=1}^{n} g(t_i^*)\Delta t$ for the function

$$g(t) := \sqrt{\big(f'(t)\big)^2 + 1}.$$

Since f' is continuous—it is *here* that I need the second additional assumption—so is g. Hence, the theory of integration guarantees that we can take the limit of the approximation:

$$\text{length} = \lim_{n\to\infty} (\text{approximation}) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + 1} \cdot \Delta t \right) = \int_a^b \sqrt{\big(f'(t)\big)^2 + 1}\, dt.$$