Exponential Solutions to $ay'' + by' + cy = 0$.

In his treatment of the differential equation

$$ay'' + by' + cy = 0 \ (a, b, c \text{ constants}), \tag{1}$$

the author suggests looking for a solution of the form $y = e^{rt}$ because a solution to (1) would would have to be of the "same type" as its first and second derivative in order for everything to cancel out. In this handout, I will offer what is to me a more compelling reason. To simplify the discussion, I will consider a specific example, but it should be clear that the considerations apply to all instances of (1).

I will examine the linear operator

$$L[y] = y'' + y' - 6y, \tag{2}$$

which gives rise to the differential equation

$$L[y] = y'' + y' - 6y = 0. \tag{3}$$

The characteristic polynomial[1] of L is the polynomial

$$p(r) = r^2 + r - 6 = (r+3)(r-2), \tag{4}$$

and the factors $(r+3)$ and $(r-2)$ are themselves the characteristic polynomials, respectively, of the linear operators

$$\begin{cases} L_1[y] = y' + 3y \\ \text{and} \\ L_2[y] = y' - 2y. \end{cases}$$

The first important observation is this:

Proposition 1. *For any function y with continuous second derivative, $L[y] = L_1\big[L_2[y]\big] = L_2\big[L_1[y]\big]$.*

Proof. By routine computation,

$$\begin{aligned} L_1\big[L_2[y]\big] &= L_1[y' - 2y] \\ &= \big(y' - 2y\big)' + 3\big(y' - 2y\big) \\ &= y'' - 2y' + 3y' - 6y \\ &= y'' + y' - 6y \\ &= L[y]; \end{aligned}$$

a precisely similar computation shows that $L_2\big[L_1[y]\big] = L[y]$. ∎

The other observation is that

$$\begin{cases} L_1[y] = 0 \iff y' = -3y \iff y = Ce^{-3t} \\ \text{and} \\ L_2[y] = 0 \iff y' = 2y \iff y = Ce^{2t}. \end{cases}$$

It now follows that $y = e^{-3t}$ and $y = e^{2t}$ are solutions to (3), since

$$\begin{cases} L[e^{-3t}] = L_2\big[L_1[e^{-3t}]\big] = L_2[0] = 0 \\ \text{and} \\ L[e^{2t}] = L_1\big[L_2[e^{2t}]\big] = L_1[0] = 0. \end{cases}$$

---

[1] It makes sense to work with the polynomial $p(r)$ rather than the equation $p(r) = 0$ because we will be dealing principally with linear operator (2) not equation (3).