The "Chart Method" for Computing Certain Integrals

The integrals I want to consider are of the form

$$\int p(x)g(x)\,dx, \tag{1}$$

where p is a polynomial and g is easy to integrate repeatedly.[1] Such an integral can be computed by repeated integration by parts, as you have already seen; the number of IBPs you need is equal to the degree of p. But, as you have also seen, repeated IBPs are long and messy. The method I am calling the "chart method" gives you a relatively easy computational way of computing (1) by predicting the final outcome of the IBPs.

To explain the method, I need names for the integrals of g. Let

$$g_1(x) = \int g(x)\,dx; \quad g_2(x) = \int g_1(x)\,dx; \quad g_3(x) = \int g_2(x)\,dx; \quad \textit{etc.}$$

Now, here is the method. Make a chart with three columns,

$p(x)$	$g_1(x)$	1
$p'(x)$	$g_2(x)$	-1
$p''(x)$	$g_3(x)$	1
$p'''(x)$	$g_4(x)$	-1
$\vdots$	$\vdots$	$\vdots$

continuing until you're up to the last nonzero derivative of p. (The last column is always alternating ± 1.) Then, to compute (1), you simply multiply across each column and add the results:

$$\int p(x)g(x)\,dx = p(x)g_1(x) - p'(x)g_2(x) + p''(x)g_3(x) - p'''(x)g_4(x) + \cdots.$$

For example, let's use the method to find $\int (x^3 + x)\cos(2x)\,dx$. Fill in the chart one column at a time:

$x^3 + x$	$\frac{1}{2}\sin(2x)$	1
$3x^2 + 1$	$-\frac{1}{4}\cos(2x)$	-1
$6x$	$-\frac{1}{8}\sin(2x)$	1
6	$\frac{1}{16}\cos(2x)$	-1

Then, using the chart, write down the answer:

$$\int (x^3 + x)\cos(2x)\,dx = \frac{1}{2}(x^3 + x)\sin(2x) + \frac{1}{4}(3x^2 + 1)\cos(2x) - \frac{1}{8}(6x)\sin(2x) - \frac{1}{16}(6)\cos(2x) + C.$$

[1] In fact, the chart method is right so long as p is a polynomial—regardless whether g is easy to integrate or not. However, if you can't integrate g repeatedly, the method is of no practical use.