

### Equality of Mixed Partial (Clairaut's Theorem)

In chapter 14, we encountered *Clairaut's Theorem*, which says that when  $f_{xy}$  and  $f_{yx}$  are both continuous, they will be equal. This was presented as a *fact*—something to believe only because I told you so. Such teaching by dictatorship is always regrettable; a guiding principle of the discipline of mathematics is that it is democratic and egalitarian—that anyone can challenge anyone else's statement, and logic will judge who is right. I had no choice, though; the justification of Clairaut's Theorem would not have made sense to you at the time.

It is now possible to close this gap: using the perspective of § 15.2, I can now back up the claim of Clairaut's Theorem with a proof. I will break off the main part as a lemma.

**Lemma.** *Let  $f_{xy}$  and  $f_{yx}$  be continuous on rectangle  $R = [a, b] \times [c, d]$ . Then*

$$\iint_R f_{xy} dA = \iint_R f_{yx} dA = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

*Proof.* I will prove the assertion for  $f_{xy}$ ; the proof for  $f_{yx}$  will be left as an exercise.

$$\begin{aligned} \iint_R f_{xy} dA &= \int_a^b \left( \int_c^d f_{xy}(x, y) dy \right) dx \\ \text{(by the FTC)} \longrightarrow &= \int_a^b \left( f_x(x, y) \Big|_{y=c}^{y=d} \right) dx \\ &= \int_a^b f_x(x, d) - f_x(x, c) dx \\ &= \int_a^b f_x(x, d) dx - \int_a^b f_x(x, c) dx \\ \text{(by the FTC)} \longrightarrow &= f(b, d) - f(a, d) - (f(b, c) - f(a, c)) \\ &= f(b, d) - f(a, d) - f(b, c) + f(a, c). \end{aligned}$$

**Exercise.** Prove that

$$\iint_R f_{yx} dA = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

This will complete the proof of the lemma. ■

**Theorem.** *If  $f_{xy}$  and  $f_{yx}$  are continuous, then they are equal.*

*Proof by contradiction.* Suppose they are not identically equal. Then at some point  $(a, b)$ , they differ; say

$$f_{xy}(a, b) - f_{yx}(a, b) = h > 0.$$

Then by continuity, there is some small  $\Delta x \times \Delta y$  rectangle, centered at  $(a, b)$ , on which

$$f_{xy}(x, y) - f_{yx}(x, y) \geq \frac{h}{2},$$

so that

$$\iint_R f_{xy} - f_{yx} dA \geq \iint_R \frac{h}{2} dA = \frac{h}{2}(\Delta x)(\Delta y) > 0.$$

But this contradicts the lemma: since  $f_{xy}$  and  $f_{yx}$  have equal integrals over the rectangle, necessarily also

$$\iint_R f_{xy} - f_{yx} dA = \iint_R f_{xy} dA - \iint_R f_{yx} dA = 0. \quad \blacksquare$$