An Introduction to the Complex Numbers

I. Heuristic beginnings.

In many mathematical contexts, it would be convenient to have a number system that includes a square root of (-1). To achieve this end, mathematicians began[1] by just *defining* such a number into existence and seeing how far they could get with it. I will follow that approach in this section. Let i be—well, *something*—with the property that

$$i^2 = -1.$$

Obviously, this i, whatever it is, will not be an ordinary real number, because no real number has this property. A number system containing this i would have to allow addition, subtraction, multiplication and division of its numbers; so it would also have to include the products bi for each real b, and then the sums $a + bi$ for real numbers a and b. After one allows just these particular products and sums, two things turn out to happen:

Theorem 1. [i], *There are no repeats among the numbers* $\{a + bi\colon a, b \text{ real}\}$*: in other words,*

$$a + bi = c + di \Longrightarrow a = c \text{ and } b = d.$$

[ii], *The set* $\{a + bi : a, b \text{ real}\}$ *is closed under addition, under subtraction, under multiplication, and under division by elements other than zero.*

Proof. [i]: Suppose $a + bi = c + di$ for real numbers a, b, c, and d. Then by algebra,

$$a - c = (d - b)i. \tag{$*$}$$

If $d - b \neq 0$, then we could solve for i:

$$i = \frac{a - c}{d - b},$$

which would make i a real number, contradiction. So $(d - b) = 0$, and hence, in $(*)$, $(a - c) = 0$ as well.

[ii]: Using ordinary algebra and the equation $i^2 = (-1)$, it is easy to verify that

$$(a + bi) \pm (c + di) = a \pm c + (b \pm d)i;$$
$$(a + bi) \cdot (c + di) = ac - bd + (bc + ad)i;$$

and, if $c + di \neq 0$,[2]

$$(c + di) \cdot \left(\frac{c}{c^2 + d^2} - \left(\frac{d}{c^2 + d^2} \right) i \right) = 1,$$

so that $c + di$ has a reciprocal:

$$\frac{1}{c + di} = \frac{c}{c^2 + d^2} - \left(\frac{d}{c^2 + d^2} \right) i. \blacksquare$$

II. The Formal Definition.

The section above shows a little about what this number system with an "i" (called the *system of complex numbers*) must be like, *provided that it exists at all*. The computations in the first section do not guarantee this; after all, seat-of-the-pants computations with a new symbol are not enough to guarantee that there really is a consistent complex number system where everything works. We need and still lack a precise definition of the complex number system as a particular algebraic system: a clearly defined set with a clearly

[1] Well, it is plausible that they may have done this, but I don't know for a fact that they actually did. I have done no research into the actual history of the development of this subject.

[2] I explain in §**IV** below where this mysterious formula comes from.

defined addition and a clearly defined multiplication. We will accomplish this by building on what we already have: the real numbers (**R**). Mathematicians have a method for building **Z** from **N**, another for building **Q** from **Z**, and a third one for building **R** from **Q**; thus, number systems **Z**, **Q**, and **R** are not in doubt. What is needed here is the fourth step: the method mathematicians have for constructing of the set of complex numbers **C**, starting from **R**.

The heuristics above indicate how to do this. The fact that

$$a + bi = c + di \qquad \Longleftrightarrow \qquad a = c \text{ and } b = d$$

is structurally identical to the fact that for points in the plane

$$(a, b) = (c, d) \qquad \Longleftrightarrow \qquad a = c \text{ and } b = d,$$

which suggests identifying each complex number "$a + bi$" as the point (a, b) in the plane; that is, as a set, **C** is just defined to be $\mathbf{R} \times \mathbf{R}$. As for the operations: the computation

$$(a + bi) \pm (c + di) = a \pm c + (b \pm d)i$$

suggests defining

$$(a, b) + (c, d) := (a + c, b + d), \tag{1}$$

and the computation

$$(a + bi) \cdot (c + di) = ac - bd + (bc + ad)i$$

suggests defining

$$(a, b) \cdot (c, d) := (ac - bd, bc + ad). \tag{2}$$

This is in fact what one does. The system of complex numbers is realized as the set of points in the plane, with the addition and multiplication defined in (1) and (2) above. Then, formal versions of the computations in part [*ii*] of Theorem 1 (and a few more computations to demonstrate things like commutativity and associativity) prove the following theorem:

Theorem 2. [*i*]: *The set* $\mathbf{C} = \mathbf{R} \times \mathbf{R}$, *together with the addition defined in (1) is an abelian group;* [*ii*]: *The set* $\mathbf{R} \times \mathbf{R} - \{(\mathbf{0}, \mathbf{0})\}$, *together with and multiplication defined in (2), is an abelian group;* [*iii*]: *The multiplication distributes over the addition.* ∎

(Such an algebraic structure is called a *field*.)

Outline of Proof. As an additive group, **C** is the Abelian group $\mathbf{R} \oplus \mathbf{R}$. By routine computations, one easily checks that multiplication is commutative and associative; that multiplication distributes over addition; that the multiplicative identity is $(1, 0)$; and that the multiplicative inverse of $(a, b) \neq (0, 0)$ is $\left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2}\right)$. ∎

III. Removing the quotes from "$a + bi$".

We now have the point (a, b) representing the the complex number "$a + bi$", but the but the plus sign and the implied multiplication in the expression "$a + bi$" are not instances of the addition and multiplication defined on $\mathbf{R} \times \mathbf{R}$ in (1) and (2). It will be very convenient to rectify this situation, and we can do so. First: it is easy to see that the set of points $\{(a, 0) : a \text{ in } \mathbf{R}\}$—that is, the x–axis—behaves exactly the way the set of real numbers behaves (exercise **1** below). So we identify **R** with the x-axis of **C** (henceforth: the "*real axis*") and simply write "a" for the complex number $(a, 0)$.

Second: because $(0, 1)$ represents "$0 + 1i$," we *define*

$$i := (0, 1);$$

that is, from now on the letter i stands for the point $(0,1)$. As you would hope, the square of this complex number is indeed (-1) (exercise **2**). Finally, having identified entities "a" "b" and "i" as points in the plane, we can verify that (a,b) does indeed equal $a+b\cdot i$ (exercise **3**), so that we may treat "$a+bi$" as an arithmetic expression and not just a name for the point (a,b).

Exercise 1: Show, for any real numbers a and b: **[i]**, $(a,0)+(b,0)=(a+b,0)$; **[ii]**, $(a,0)\cdot(b,0)=(ab,0)$.

Exercise 2: Show that

$$(0,1)\cdot(0,1)=(-1,0);$$

that is, that $i^2=(-1)$.

Exercise 3: Show for any (a,b) in **C**, that

$$(a,b)=(a,0)+(b,0)\cdot(0,1);$$

that is, that $(a,b)=a+b\cdot i$.
(For $z=a+bi$, the real numbers a [or $(a,0)$] and b [or $(b,0)$] are respectively called the *real* and *imaginary* parts of z.)

IV. Modulus and Conjugate.

For each complex number $z=a+bi$, one defines the *modulus* (or *norm*) of z to be the number[3]

$$|z|:=\sqrt{a^2+b^2}$$

and the *complex conjugate* of z to be the number

$$\overline{z}:=a-bi.$$

Obviously, $|z|$ is the distance from z to the origin, and $\overline{z}$ is the reflection of z in the real axis. Obviously also, $\overline{z}=z$ if and only if z is a real number (that is, if and only if $b=0$).

The following facts are proved by follow-your-nose, straighforward computations.

Exercise 4: Prove:

[a], $\overline{z+w}=\overline{z}+\overline{w}$.

[b], $\overline{zw}=\overline{z}\,\overline{w}$.

[c], $z\overline{z}=|z|^2$.

I can now explain the origin of the mysterious formula for $\dfrac{1}{c+di}$ (see p.1). Let $z=(c,d)\neq(0,0)$ in **C**. Then

$$\begin{aligned} 1 &= \frac{|z|^2}{|z|^2} \\ \text{exercise 4[c]} \rightarrow &= \frac{z\overline{z}}{|z|^2} \\ \text{regroup} \rightarrow &= z\left(\frac{\overline{z}}{|z|^2}\right). \end{aligned}$$

Therefore, the multiplicative inverse of z is the thing in the parentheses:

$$\frac{1}{z}=\frac{\overline{z}}{|z|^2}=\frac{c}{|z|^2}-\frac{di}{|z|^2}.$$

I am also in a position to demonstrate the first of the two key facts that explain how to understand complex multiplication geometrically.

[3] $|z|$ is a nonnegative real number, but keep in mind that we have identified it with the complex number $(|z|,0)$, since **R** has been identified with the real axis in **C**.

Theorem 3. *For any complex numbers z and w,*

$$|zw| = |z| \cdot |w| \, .$$

Proof. I will prove that both of these numbers have the same square; since they're both nonnegative real numbers, it will follow that they're the same number.

$$\begin{aligned} |zw|^2 &= \\ \text{exercise 4[c]} \to &= zw\overline{zw} \\ \text{exercise 4[b]} \to &= zw\overline{z}\,\overline{w} \\ &= z\overline{z}w\overline{w} \\ \text{exercise 4[c]} \to &= |z|^2 \, |w|^2 \, . \quad \blacksquare \end{aligned}$$

V. The Geometric Meaning of Multiplication.

As noted above, Theorem 3 tells you that when you multiply complex numbers z and w, you multiply their moduli; in other words, for nonzero complex numbers z and w, zw will lie somewhere on the circle of radius $|z|\,|w|$ around the origin. The other geometric fact about multiplication tells you exactly where on this circle zw will lie. It says that you *add the angles* that z and w (as radii) make with the positive real axis to find the angle that zw makes with the positive real axis. So, for example: suppose $|z| = 2$, and z makes a 30° angle with the positive x–axis; and suppose and $|w| = 3$, and w makes a 45° angle with the positive x–axis. Then zw is on the circle of radius 6 centered at the origin; and the radius from zw to the origin will make that makes an angle of 75° with the positive x–axis.

This fact can be proved by doing exercises **5–7**, which are easy but require some knowledge of trigonometry. Throughout, assume z and w are nonzero complex numbers; the first exercise establishes a useful way to factor z.

Exercise 5: If the segment from 0 (the origin) to z makes an angle of θ with the positive real axis, then

$$z = |z| \left(\cos(\theta) + i\sin(\theta)\right). \tag{3}$$

(*Hint*: First show that $\dfrac{z}{|z|}$ has modulus 1; then use trig.)

Observe that the right side of equation (3) gives the polar form of z (thought of as a vector): θ is the polar angle, and $r = |z|$.

In exercise **6**, you will prove that if you multiply two complex numbers z and w of modulus 1, you find the product by adding the angles that z and w make with the positive real axis.

Exercise 6: For any real numbers θ and ϕ,

$$(\cos(\theta) + i\sin(\theta)) \cdot (\cos(\phi) + i\sin(\phi)) = \cos(\theta + \phi) + i\sin(\theta + \phi). \tag{4}$$

(*Hint*: Multiply out, gather real and imaginary parts of the product, and apply some familiar trig identities.)

Exercise 7: Show: if z makes angle θ with the positive real axis and w makes angle ϕ with the positive real axis, then zw makes angle $\theta + \phi$ with the positive real axis.

(*Hint*: Use exercises **4b**, **5** and **6**.)

VI. The Complex Exponential Function.

For a real number θ, it turns out to be the right move[4] to define

$$e^{i\theta} := \cos(\theta) + i\sin(\theta),$$

for a number of reasons, of which I will discuss three. First, equation (3) can be rewritten

$$z = |z|e^{i\theta}, \tag{3'}$$

which is useful both conceptually and computationally. Second, making this substitution (three times) in equation (4) reveals it to be an exponential law:

$$\begin{aligned} e^{i\theta} \cdot e^{i\phi} &= e^{i(\theta+\phi)} \\ &= e^{i\theta+i\phi}. \end{aligned} \tag{4'}$$

Third, if for any $z = a + bi$, you define the exponential e^z by

$$e^z := e^a e^{bi}, \tag{5}$$

then it turns out that you get the full exponental law

$$e^{z+w} = e^z e^w.$$

Observe that in equation (5), $e^a e^{bi}$ is in polar form (see remark after exercise **5**).

Exercise 8: Show, for any complex numbers z and w, that $e^{z+w} = e^z e^w$.

VII. The Derivative of $t \mapsto e^{it}$.

We will have need of this derivative; to determine what it is, I will first discuss the general question of how to define the derivative of a function $f(t) = u(t) + iv(t)$, where u and v are differentiable real-valued functions of t. The natural plausible definition is

$$f'(t) := \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}, \tag{6}$$

where the subtraction, the division, and the limit are all computed in the complex plane. Fortunately, this limit turns out to be an old friend, from Calc III. Since the only division is by the real number h, one can reinterpret the subtraction as vector subtraction and the division as scalar multiplication by $1/h$ (in $\mathbf{R}^2$). This recasts f as a vector-valued function $\vec{f}(t) = \langle u(t), v(t)\rangle$, and limit (6), written in this language, becomes

$$\lim_{h\to 0} \frac{1}{h}\left[\vec{f}(t+h) - \vec{f}(t)\right]. \tag{7}$$

Now, (7) is the definition of the derivative of $t \mapsto \vec{f}(t)$ (see Stewart p.824), and the first theorem one proves about this derivative is that it can be computed coordinatewise:

$$\lim_{h\to 0} \frac{1}{h}\left[\vec{f}(t+h) - \vec{f}(t)\right] = \langle u'(t), v'(t)\rangle.$$

This gives us the computational form of (6):

$$f'(t) := \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \langle u'(t), v'(t)\rangle. \tag{6'}$$

(The text takes the computational form as the definition of $f'(t)$ (Braun p.141). It is important to note that using this form, Braun goes on to prove Lemma 1.)

[4] Consideration of the Maclaurin series of $x \mapsto e^x$ can also lead one to this definition and to that in equation (5).

The answer to the main question is now immediate. Applying (6′) to the function $f(t) = e^{it}$ gives

Theorem 4.

$$\frac{d}{dt}\left[e^{it}\right] = ie^{it}. \blacksquare$$

Exercise 9: Prove Theorem 4.

VIII. A Glimpse down the Road.

This is a deep and beautiful subject that holds many astonishments. I want to mention only one fact, which concerns the question: will we ever want to extend the set of numbers even further, beyond **C**? **C** is the result when we extend **R** by throwing in a solution to the equation

$$x^2 + 1 = 0;$$

will we ever extend **C** to some larger system by throwing in a solution to some other polynomial equation? The answer to this question turns out to be *no*: all solutions to all polyomial equations are already present in **C**. The precise theorem, proved originally by Gauss, is:

Theorem 5 (Fundamental Theorem of Algebra). *Every polynomial of degree $n \geq 1$ with complex coefficients will have (counting multiple roots) exactly n solutions in* **C**.

Thus, for example, you will not need to extend **C** to find the fourth roots of (-1) (the solutions to $x^4+1 = 0$). They're already there.

Exercise 10: Using the geometric interpretation of multiplication, find a complex number z such that $z^4 = -1$.[5]

[5] Actually, there are *four* of them. Can you find them all??