

### What Goes Wrong with the Deduction Theorem in $\mathcal{K}$

As will be discussed in class, there are certain subsidiary deductions  $\Gamma \oplus [\mathcal{A}] \vdash_{\mathcal{K}} \mathcal{B}$  to which there do not correspond deductions  $\Gamma \vdash_{\mathcal{K}} (\mathcal{A} \rightarrow \mathcal{B})$ . Here is an example to illustrate why a further restriction is necessary.

Consider the wf

$$\mathcal{A} : (A_1^2(x_1, a_1) \rightarrow A_1^2(f_1^1(a_1), a_1));$$

in the extension of  $\mathcal{K}$  that axiomatizes (part of) the system of natural numbers, this wf has the nickname

$$((x_1 = 0) \rightarrow (1 = 0)). \tag{1}$$

When interpreted, this wf could mean several things; observe that only meaning (a) is logically valid:

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|--|---|
| (a) “If all numbers $x$ equal zero, then $1 = 0$ .”                              | $((\forall x_1)(x_1 = 0) \rightarrow (1 = 0))$      |
| (b) “If the number named by $x_1$ equals 0, then $1 = 0$ .”                      | Either $v(x_1) \neq \bar{0}$ or $\bar{1} = \bar{0}$ |
| (c) “For all numbers $x$ , the implication ‘ $x = 0 \implies 1 = 0$ ’ is valid.” | $(\forall x_1)((x_1 = 0) \rightarrow (1 = 0))$      |

Now, there is a subsidiary deduction  $x_1 = 0 \vdash_{\mathcal{K}} 1 = 0$ :

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|----|--|------------|
| 1. | $x_1 = 0$                                    | assumption |
| 2. | $(\forall x_1)(x_1 = 0)$                     | gen(1)     |
| 3. | $(\forall x_1)(x_1 = 0) \rightarrow (1 = 0)$ | K5         |
| 4. | $1 = 0$                                      | MP(2,3)    |

Observe that in line (2), the subsidiary deduction is forced—as it should be—to declare explicitly that it is assigning the meaning “All numbers equal zero” to the wf  $x_1 = 0$ . A wf returned by a valid Deduction Theorem would therefore need to declare this as well; in other words, the wf would have to declare its meaning to be the logically valid (a). Since wf (1) has no such explicit declaration built into it, so that (b) and (c) are also possible interpretations of it, wf (1) fails, informally at least, to be logically valid. It is thus probably not a theorem of  $\mathcal{K}$ . (In Exercise (1) you will show that wf (1) is in fact *formally* not logically valid, so it *definitely* fails to be logic be a theorem of  $\mathcal{K}$ .)

**Exercise 1.** Show that (1) is (formally) not logically valid.

Let me push the example further. Suppose we were mistakenly to write down:

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|----|---------------------------------|-------------------|
| 1. | $(x_1 = 0) \rightarrow (1 = 0)$ | Deduction Theorem |
|----|---------------------------------|-------------------|

We could follow this with:

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|----|--|--------|
| 2. | $(\forall x_1)((x_1 = 0) \rightarrow (1 = 0))$ | gen(1) |
|----|--|--------|

(Observe that this move has assigned meaning (c) to wf (1).) Then, the following further moves would be possible:

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|----|--|---------|
| 3. | $((\forall x_1)((x_1 = 0) \rightarrow (1 = 0))) \rightarrow ((0 = 0) \rightarrow (1 = 0))$ | K5      |
| 4. | $(0 = 0) \rightarrow (1 = 0)$  | MP(2,3) |

The wfs 2 and 4, like (1), are clearly not logically valid, and so are not theorems of  $\mathcal{K}$ . Indeed, in the extension of  $\mathcal{K}$  that embodies the natural numbers, both  $0 = 0$  and  $\sim(1 = 0)$  turn out to be theorems, as they should; so the natural-number robot would then be able to continue with the moves:

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|----|---------------|---------|
| 5. | $0 = 0$       | theorem |
| 6. | $1 = 0$       | MP(5,4) |
| 7. | $\sim(1 = 0)$ | theorem |

We thus would have caught the natural-number robot, *flagrante delicto*, generating a contradiction! But not to worry: wf (1) is not a theorem of  $\mathcal{K}$ ; as you will learn, the valid Deduction Theorem cannot be applied to the subsidiary deduction above.