Introduction to Numeralwise Expressibility

Consider the $\mathcal{N}$–*wf*

$$\mathcal{G}(x_1, x_2) := (\exists x_3)(x_3' + x_1 = x_2).$$

Clearly, when interpreted in the standard way, $\mathcal{G}(x_1, x_2)$ means "x_1 is less than x_2;" furthermore, it is easy to check, for any valuation v in the usual interpretation **N** of $\mathcal{N}$,[1] that

$$f_v\left(G(x_1, x_2)\right) = T \iff v(x_1) < v(x_2).$$

Thus, the meaningful predicate "x_1 is less than x_2," expresses the $\mathcal{N}$–*wf* $\mathcal{G}(x_1, x_2)$ in a certain sense. An important relationship between $\mathcal{N}$ and **N**—one that is crucial for Gödel's theorems—is that often this expressibility can be reversed: in a certain sense (to be made precise later), the truth or falseness of a meaningful predicate in **N** will guarantee proofs of corresponding theorems in $\mathcal{N}$. This handout will first establish this sort of "reverse expressibility" between "x_1 is less than x_2" and $\mathcal{G}(x_1, x_2)$; the general definition will follow.

Lemma 1. *For all nonnegative integers k and ℓ,*

$$\vdash_{\mathcal{N}} 0^k + 0^\ell = 0^{k+\ell}.$$

Outline of proof. The proof proceeds by induction on ℓ, although I will not write this out formally. Start with the observation that by using $\mathcal{K}5$ $N3^*$ and MP, one can clearly get $\vdash_{\mathcal{N}} \left(0^k + 0 = 0^k\right)$. deduce from this fact the fact that $\vdash_{\mathcal{N}} 0^k + 0' = 0^{k+1}$, as follows.

$\vdash_{\mathcal{N}} 0^k + 0 = 0^k$ (as mentioned above)

$\vdash_{\mathcal{N}} 0^k + 0' = (0^k + 0)'$ ($\mathcal{K}5$ applied to $N4^*$)

$\vdash_{\mathcal{N}} 0^k + 0 = 0^k \to (0^k + 0)' = (0^k)'$ ($E8^*$)

$\vdash_{\mathcal{N}} (0^k + 0)' = (0^k)'$ (MP)

$\vdash_{\mathcal{N}} 0^k + 0' = 0^{k+1}$ transitivity of $=$ (Prop 5.4 p 108); definition of 0^{k+1}.

By iterating the process, one can show that $\vdash_{\mathcal{N}} 0^k + 0'' = 0^{k+2}$, $\vdash_{\mathcal{N}} 0^k + 0''' = 0^{k+3}$, *etc.* After doing this ℓ times, you'll have $\vdash_{\mathcal{N}} 0^k + 0^\ell = 0^{k+\ell}$. ∎

Exercise 1. Using the fact that $\vdash_{\mathcal{N}} 0^k + 0' = 0^{k+1}$, show that $\vdash_{\mathcal{N}} 0^k + 0'' = 0^{k+2}$.

Lemma 2. *Let k and ℓ be nonnegative integers. If $k < \ell$, then*

$$\vdash_{\mathcal{N}} \mathcal{G}\left(0^k, 0^\ell\right).$$

Proof. First, we apply Lemma 1 to $\ell - k$ and k to obtain

$$\vdash_{\mathcal{N}} 0^{\ell-k} + 0^k = 0^\ell. \qquad (*)$$

Next, since $\ell - k \geq 1$, $(*)$ can be rewritten

$$\vdash_{\mathcal{N}} \left(0^{\ell-k-1}\right)' + 0^k = 0^\ell. \qquad (\dagger)$$

It is then a short step to

$$\vdash_{\mathcal{N}} (\exists x_3)(x_3' + 0^k = 0^\ell). \ ∎$$

Exercise 2. If t is free for x in $\mathcal{A}(x)$, show that $\vdash_{K} \mathcal{A}(t) \to (\exists x)\mathcal{A}(x)$.

Exercise 3. Use ($\dagger$) and result of Exercise (2) to supply the "short step" that finishes the proof of Lemma 2.

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[1] Text, page 57.

Lemma 3. *Let k and ℓ be nonnegative integers. If $k \geq \ell$, then*

$$\underset{\mathcal{N}}{\vdash} \sim\mathcal{G}\left(0^k, 0^\ell\right).$$

Proof. Put

$$d := k - \ell \geq 0.$$

I will divide the proof into several steps.

Step 1. $\underset{\mathcal{N}}{\vdash} x_3' + 0^d = x_3^{d+1}$.

Proof of Step 1 has exactly the same structure as the proof of Lemma 1; the details will be left to you (see Exercise 4). ∎

Exercise 4. Imitate the proof of Lemma 1 to supply the details of the proof of Step 1. (Supply about the same amount of detail as I did in that proof.)

Step 2. $x_3' + 0^k = 0^\ell \underset{\mathcal{N}}{\vdash} x_3' + 0^d = 0$.

Proof of Step 2. If $\ell = 0$ (so that $k = d$), the assumption *wf* matches the *wf* to be derived, so the assertion is obvious. If $\ell > 0$, then one can show that $x_3' + 0^k = 0^\ell \underset{\mathcal{N}}{\vdash} x_3' + 0^{k-1} = 0^{\ell-1}$, as follows.

$$\begin{array}{ll}
x_3' + \left(0^{k-1}\right)' = \left(0^{\ell-1}\right)' & \text{(assumption)} \\
x_3' + \left(0^{k-1}\right)' = \left(x_3' + 0^{k-1}\right)' & (\mathcal{K}5 \text{ applied to } N4^*) \\
\left(x_3' + 0^{k-1}\right)' = \left(0^{\ell-1}\right)' & \text{(manipulations using equality axioms)} \\
\left(x_3' + 0^{k-1}\right)' = \left(0^{\ell-1}\right)' \longrightarrow x_3' + 0^{k-1} = 0^{\ell-1} & (\mathcal{K}5 \text{ applied to } N2^*) \\
x_3' + 0^{k-1} = 0^{\ell-1} & \text{(MP)}
\end{array}$$

Iterating this process $\ell - 1$ more times will lead to a derivation of $x_3' + 0^d = 0$. ∎

Step 3. $x_3' + 0^k = 0^\ell \underset{\mathcal{N}}{\vdash} x_3^{d+1} = 0$.

Proof of Step 3: Combine the results of the first two steps and manipulate appropriately using equality axioms. ∎

Exercise 5. Supply the details. You may be vague on the equaltity axiom manipulations.

Step 4. $\underset{\mathcal{N}}{\vdash} x_3' + 0^k = 0^\ell \longrightarrow \sim(\forall x_1)(\sim(x_1' = 0))$.

$$\begin{array}{ll}
\underset{\mathcal{N}}{\vdash} x_3' + 0^k = 0^\ell \longrightarrow \left(x_3^d\right)' = 0 & \text{(DT applied to Step 3)} \\
\underset{\mathcal{N}}{\vdash} \left(x_3^d\right)' = 0 \longrightarrow \sim(\forall x_1)(\sim(x_1' = 0)) & \text{(instance of Exercise 2)} \\
\underset{\mathcal{N}}{\vdash} x_3' + 0^k = 0^\ell \longrightarrow \sim(\forall x_1)(\sim(x_1' = 0)) & \text{(HS)}
\end{array}$$

Step 5. $\underset{\mathcal{N}}{\vdash} x_3' + 0^k = 0^\ell \longrightarrow (\forall x_1)(\sim(x_1' = 0))$.

Proof: Left to you (see Exercise 6). ∎

Exercise 6. Supply the details. You will need axioms $N1^*$ and $\mathcal{K}1$ (and MP).

Step 6. $\underset{\mathcal{N}}{\vdash} \sim\mathcal{G}\left(0^k, 0^\ell\right)$.

Proof of Step 6. By applying an appropriate tautology and MP to the two $\mathcal{N}$–theorems derived in steps 4 and 5, one obtains

$$\underset{\mathcal{N}}{\vdash} \sim(x_3' + 0^k = 0^\ell),$$

from which, by generalization, one obtains

$$\underset{\mathcal{N}}{\vdash} (\forall x_3)\left(\sim(x_3' + 0^k = 0^\ell)\right).$$

Then, another tautology and MP lead to

$$\underset{\mathcal{N}}{\vdash} \sim\sim(\forall x_3)\left(\sim(x_3' + 0^k = 0^\ell)\right),$$

and this last *wf* is precisely $\underset{\mathcal{N}}{\vdash} \sim\mathcal{G}\left(0^k, 0^\ell\right)$. ∎

Exercise 7. Which tautologies are used in Step 6, and how?

The proof of Lemma 3 is complete. ∎

Numeralwise expressibility. Lemmas 2 and 3 establish a certain precise relationship between the meaningful **N**-predicate

$$P(x_1, x_2) := x_1 \text{ is less than } x_2$$

and the $\mathcal{N}$–*wf* $\mathcal{G}(x_1, x_2)$, namely that for any nonnegative integers k and ℓ : if $P(k, \ell)$ is true, then $\underset{\mathcal{N}}{\vdash} \mathcal{G}\left(0^k, 0^\ell\right)$; and if $P(k, \ell)$ is false, then $\underset{\mathcal{N}}{\vdash} \sim\mathcal{G}\left(0^k, 0^\ell\right)$. One conveys the existence of this relationship by saying that $\mathcal{G}(x_1, x_2)$ *numeralwise expresses* $P(x_1, x_2)$. Here is the general definition of which this is an instance:

Definition. *Let $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ be k variables in $\mathcal{N}$; let $\mathcal{A} = \mathcal{A}(\mathbf{y}_1, \ldots, \mathbf{y}_k)$ be an $\mathcal{N}$–wf which has exactly these k variables free—these k and no others—and let $P(x_1, \ldots, x_k)$ be a meaningful* **N***–predicate in the k indeterminates $\{x_1, \ldots, x_k\}$. One says that $\mathcal{A}$* **numeralwise expresses** *P if and only if, for all choices $(n_1, \ldots, n_k)$ of nonnegative integers,*

$$\begin{cases} P(n_1, \ldots, n_k) \text{ is true} \implies \underset{\mathcal{N}}{\vdash} \mathcal{A}(0^{n_1}, \ldots, 0^{n_k}) \\ \text{and} \\ P(n_1, \ldots, n_k) \text{ is false} \implies \underset{\mathcal{N}}{\vdash} \sim\mathcal{A}(0^{n_1}, \ldots, 0^{n_k}). \end{cases}$$