The Area of a Frustum

Frustum is the official term for the shape of an ordinary lampshade. We will have need of a formula to compute the area of a frustum; I will develop the formula in this handout. I will divide the development into four steps.

Step 1. *Let a sector of circumferential length c be cut from a circle of radius a. The area of the sector is $\frac{1}{2}ca$.*

Proof. Since the entire circumference is of length $2\pi a$,

$$\text{area of sector} = \frac{c}{2\pi a} \times \text{area of circle} = \frac{c}{2\pi a} \times \pi a^2 = \frac{1}{2}ca. \quad \blacksquare$$

Step 2. *If a right circular cone has "slant length" ℓ and radius r at the base, then its area equals $\pi r \ell$.*

Proof. Slit the cone open along a straight line from the vertex to the base; open it up; and lay it flat. Since all such straight lines have length ℓ, what was the surface of the cone is now a sector of a circle of radius $a = \ell$ and circumferential length $c = 2\pi r$. Thus (by Step 1),

$$\text{Area} = \frac{1}{2}ca = \frac{1}{2}(2\pi r)(\ell) = \pi r \ell. \quad \blacksquare$$

Step 3. *Let F be the frustum you get by cutting a cone of slant length ℓ_1 and base radius r_1 away from a larger cone of slant length ℓ_2 and base radius r_2. Then the area of F is given by the formula*

$$A = \pi \ell_2 r_2 - \pi \ell_1 r_1. \tag{1}$$

Proof: Immediate from Step 2. $\blacksquare$

Finally, I want to apply a little algebra to formula (1), in order to get the area formula into a more useful form. (Below, the symbol "$\Delta\ell$" will denote the difference $\ell_2 - \ell_1$.)

Step 4. *Another formula for the area of the frustum is:*

$$A = 2\pi\left(\frac{r_1 + r_2}{2}\right) \cdot \left(\ell_2 - \ell_1\right) = 2\pi\left(\frac{r_1 + r_2}{2}\right) \cdot (\Delta\ell).$$

Proof. Picture the cone slit open; the angle between the sides is $\theta = \dfrac{2\pi r_2}{\ell_2} = \dfrac{2\pi r_1}{\ell_1}$, so that $\dfrac{\ell_2}{r_2} = \dfrac{\ell_1}{r_1} = \dfrac{2\pi}{\theta}$. Let $h := \dfrac{2\pi}{\theta}$; then $\ell_2 = r_2 h$ and $\ell_1 = r_1 h$. Make those substitutions into (1):

$$A = \pi \ell_2 r_2 - \pi \ell_1 r_1 = \pi\left(h r_2\right) r_2 - \pi\left(h r_1\right) r_1 = \pi h\left(r_2^2 - r_1^2\right) =$$

$$\pi\left(r_1 + r_2\right) h\left(r_2 - r_1\right) = 2\pi\left(\frac{r_1 + r_2}{2}\right) h\left(r_2 - r_1\right) = 2\pi\left(\frac{r_1 + r_2}{2}\right)\left(h r_2 - h r_1\right) =$$

$$2\pi\left(\frac{r_1 + r_2}{2}\right)\left(\ell_2 - \ell_1\right) = 2\pi\left(\frac{r_1 + r_2}{2}\right)\left(\Delta\ell\right). \quad \blacksquare$$