The Generalized Binomial Theorem

The ordinary Binomial Theorem states that for any nonnegative integer n,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \text{ where } \binom{n}{k} = \left\{ \begin{array}{ll} \dfrac{n(n-1)\cdots(n-k+1)}{k!} & \text{if } k \geq 1; \\ \dbinom{n}{0} = 1 & \end{array} \right\}; \qquad (1)$$

this equation is valid for any number x. Now, the definition of $\binom{n}{k}$ above makes perfect sense if n is not a nonnegative integer; so a natural question is whether Equation (1) will remain valid for other exponents. In other words: for what real numbers α besides nonnegative integers will the formula below be true?

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \qquad (2)$$

where

$$\binom{\alpha}{k} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k \geq 1; \\ \dbinom{\alpha}{0} = 1. & \end{cases}$$

It turns out that Equation (2)—the *Generalized Binomial Theorem*—is indeed true for every choice of exponent α; however, one must restrict x to the range $|x| < 1$. Here is an outline of a proof of this fact.

Let $f(x) = (1+x)^\alpha$ and let $g(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$. The assertion to be proved is that $f(x) \equiv g(x)$ for $|x| < 1$.

Step 1. *The series that defines $g(x)$ converges for $|x| < 1$.*

Proof. Use the Ratio Test.

Step 2. *The function f satisfies the differential equation*

$$(1+x)y'(x) \equiv \alpha y(x).$$

Proof. Routine computation.

Step 3. *The function g also satisfies the differential equation*

$$(1+x)y'(x) \equiv \alpha y(x).$$

Proof. Messy computation (which I may do for you in class).

Step 4. *Any two functions that satisfy this differential equation are equal up to a multiplicative constant.*

Proof. Let $y(x)$ be any such function. Show that the ratio $H(x) = \dfrac{y(x)}{f(x)}$ is constant because $H'(x) \equiv 0$.

Step 5. *The multiplicative constant relating $f(x)$ to $g(x)$ is equal to 1.*

Proof. Evaluate both functions at $x = 0$. ∎