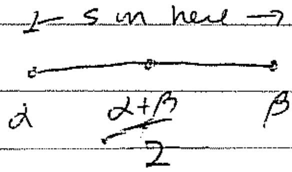
Estimating $S = \sum_{k=1}^{\infty} f(k)$ if $f(x)$ is pos, decreasing, $\int_1^{\infty} f(x)\,dx$ exists, and you can calculate $\int f(x)\,dx$. (Put $S_n := \sum_{k=1}^{n} f(k)$.)

$$\int_{n+1}^{\infty} f(x)\,dx \le S - S_n \le \int_{n}^{\infty} f(x)\,dx$$

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{\alpha} \le S \le \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{\beta}$$



Take estimate $\frac{\alpha+\beta}{2} = S_n + \frac{1}{2}\int_{n+1}^{\infty} f(x)\,dx + \frac{1}{2}\int_{n}^{\infty} f(x)\,dx$

$$|S - \text{estimate}| \le \frac{\beta-\alpha}{2} = \frac{1}{2}\int_{n}^{n+1} f(x)\,dx.$$

Example: $\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$ $\qquad \int_a^{\infty} x e^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\int_{-a^2}^{-t^2} e^{u}\,du$ $\quad (u = -x^2)$

$$= \lim_{t\to\infty} \frac{-1}{2e^{t^2}} + \frac{1}{2e^{a^2}} = \frac{1}{2e^{a^2}}$$

So $\int_1^{\infty} x e^{-x^2} = \frac{1}{2e}$. (Series converges by Int. Test.)

Let $n = 10$; $S_{10} = \sum_{k=1}^{10} \frac{k}{e^{k^2}}$

$$\text{Estimate} = S_{10} + \frac{1}{2}\int_{10}^{\infty} x e^{-x^2}\,dx + \frac{1}{2}\int_{11}^{\infty} x e^{-x^2}\,dx$$

$$= S_{10} + \frac{1}{4e^{100}} + \frac{1}{4e^{121}}$$

$$|S - S_{10}| \le \frac{1}{2}\int_{10}^{11} x e^{-x^2}\,dx = \frac{-1}{4e^{121}} + \frac{1}{4e^{100}} = \frac{1}{4e^{100}} - \frac{1}{4e^{121}}$$