

### The Adequacy Theorem for $\mathcal{K}$

The proof that every logically valid *wf* is a theorem of  $\mathcal{K}$  is similar in gross structure to the text's proof that every tautology is a theorem of  $L$ , but—surprise, surprise—it is much harder. Recall that the proof for  $L$  reasons by contradiction; it turns on finding, for  $\mathcal{A}$  that is not a theorem of  $L$ , a valuation  $v$  such that  $v(\mathcal{A}) = F$ . The corresponding proof for  $\mathcal{K}$  works by finding, for  $\mathcal{A}$  such that is not a theorem of  $\mathcal{K}$ , an interpretation  $I$  and a valuation  $v$  in  $I$  such that  $f_v(\mathcal{A}) = F$ . The text proves this as Proposition 4.41 in §4.4, but all of the hard work is done in proving Proposition 4.40. It makes sense to introduce one definition from §4.5 before launching into the details.

**Definition.** Let  $S$  be a first-order system. A *model* of  $S$  is an interpretation in which every theorem of  $S$  is true.

The following is an immediate consequence of remark 3.25c (page 63):

**Exercise 1.** Show that if  $S$  has a model, then  $S$  is consistent.

The converse is by no means so obvious, but it is true. (I believe that the ideas in the proof are due to Gödel, but I am not sure.)

**Theorem (Proposition 4.40).** *Let  $S$  be a first-order system. If  $S$  is consistent, then  $S$  has a model.*

*Proof.* I will break the proof into several steps.

**Step 1.** Form a new first-order system  $S^+$  by adjoining to  $S$  new individual constants  $\{b_0, b_1, b_2, \dots\}$ .<sup>1</sup> I **claim** that  $S^+$  is consistent.

**Proof of claim.** I will prove the contrapositive. Suppose on the contrary that there are some *wf*  $\mathcal{A}$  such that  $\vdash_{S^+} \mathcal{A}$  and  $\vdash_{S^+} \sim \mathcal{A}$ . Say that  $\{b_{k_1}, \dots, b_{k_n}\}$  are the new constants that appear in these deductions. Find variables  $\{y_{k_1}, \dots, y_{k_n}\}$  that do not appear anywhere in the deductions; and throughout both deductions and for each  $1 \leq i \leq n$ , replace each occurrence of  $b_{k_i}$  with  $y_{k_i}$ , so as to form two new sequences of formulas. Any axiom in either deduction that contains a  $b_{k_i}$  must be an instance of  $\mathcal{K}1$ – $\mathcal{K}6$ —see footnote—and the *wf* that replaced this axiom in the new sequence is another valid instance of the same axiom. All other axioms came over unchanged. Furthermore, any applications of MP (respectively gen) in the original deductions correspond to valid applications of MP (respectively gen) in the new sequences. The new sequences are thus valid deductions in  $S$  of two contradictory formulas, so that  $S$  is not consistent. ■(claim)

**Step 2.** Let  $(F_1(x_{i_1}), F_2(x_{i_2}), F_3(x_{i_3}), \dots)$  be an enumeration of all the *wfs* (of the extended language) that have exactly one free variable. (Multiple occurrences of the variable in the *wf* are permitted.) Now do each of the following. First, choose a subsequence  $(b_{j_1}, b_{j_2}, b_{j_3}, \dots)$  of the  $b$ 's such that the  $b$ 's in the sequence are distinct ( $k \neq \ell \implies b_{j_k} \neq b_{j_\ell}$ ) and such that, for each  $n \geq 1$ , the constant  $b_{j_n}$  does not appear in any of the *wfs*  $(F_1(x_{i_1}), F_2(x_{i_2}), F_3(x_{i_3}), \dots, F_n(x_{i_n}))$ . Second, let  $\mathcal{G}_n$  be the *wf* defined by

$$\mathcal{G}_n := F_n(b_{j_n}) \rightarrow (\forall x_{i_n})F_n(x_{i_n}).$$

Third, form the sequence of extensions  $(S_0, S_1, S_2, \dots)$ , where  $S_0$  is  $S^+$  and, for each  $n \geq 0$ ,  $S_{n+1}$  is  $S_n \oplus [\mathcal{G}_{n+1}]$ . I **claim** that  $S_n$  is consistent, for all  $n \geq 0$ .

**Proof of claim.** Suppose not. Then there is a smallest natural number  $n_0$  such that  $S_{n_0}$  is inconsistent. (Observe that  $n_0 \neq 0$  because  $S^+$  is consistent, by Step 1. I will get a contradiction by showing that  $n_0$  is not minimal.)

Since  $S_{n_0}$  is inconsistent, every *wf* is a theorem of  $S_{n_0}$ , including  $\sim \mathcal{G}_{n_0}$ :

$$\vdash_{S_{n_0}} \sim \mathcal{G}_{n_0},$$

so that

$$\mathcal{G}_{n_0} \vdash_{S_{n_0-1}} \sim \mathcal{G}_{n_0}.$$

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<sup>1</sup> Furthermore, except for  $\mathcal{K}1$ – $\mathcal{K}6$ , allow no axiom schemata in  $S$ , so that the only *axioms* that are new to  $S^+$  are further instances of the  $\mathcal{K}$  axioms.

Since  $\mathcal{G}_{n_0}$  is closed, one can now apply the Deduction theorem:

$$\vdash_{S_{n_0-1}} \mathcal{G}_{n_0} \rightarrow \sim \mathcal{G}_{n_0}.$$

If you append to this deduction the tautology  $(\mathcal{G}_{n_0} \rightarrow \sim \mathcal{G}_{n_0}) \rightarrow \sim \mathcal{G}_{n_0}$  and apply MP, you then have

$$\vdash_{S_{n_0-1}} \sim \mathcal{G}_{n_0}; \quad \text{that is,} \quad \vdash_{S_{n_0-1}} \sim (F_{n_0}(b_{j_{n_0}}) \rightarrow (\forall x_{i_{n_0}})F_{n_0}(x_{i_{n_0}})).$$

To this last deduction, you can then append the appropriate instances of the tautologies

$$\left\{ \begin{array}{l} ((\sim(\mathcal{A} \rightarrow \mathcal{B})) \rightarrow \mathcal{A}) \\ ((\sim(\mathcal{A} \rightarrow \mathcal{B})) \rightarrow \sim \mathcal{B}) \end{array} \right.$$

and apply MP to get

$$\vdash_{S_{n_0-1}} F_{n_0}(b_{j_{n_0}}) \tag{A}$$

and

$$\vdash_{S_{n_0-1}} (\sim(\forall x_{i_{n_0}})F_{n_0}(x_{i_{n_0}})). \tag{B}$$

Now, consider deduction (A). If you make sequence (A') from (A) by replacing each copy of  $b_{j_{n_0}}$  with a variable  $y$  which appears nowhere, free or bound, in (A), then (A') is also a valid deduction in  $S_{n_0-1}$ :

→[a] Each tautology axiom of  $\mathcal{K}$  that changes is replaced with another tautology axiom of  $\mathcal{K}$ .

→[b] Each instance of  $\mathcal{K}5$  that changes nontrivially is of the form

$$(\forall z)\mathcal{A}(z) \rightarrow \mathcal{A}(b_{j_{n_0}});$$

this will be replaced by

$$(\forall z)\mathcal{A}(z) \rightarrow \mathcal{A}(y),$$

which is again a valid instance of  $\mathcal{K}5$ , since  $y$  does not appear in  $\mathcal{A}(z)$ .

→[c] Each instance of  $\mathcal{K}6$  that changes will be changed to another valid instance of  $\mathcal{K}6$ .

**Exercise 2.** Prove [c] (above).

→[d] Each of the formulas  $\mathcal{G}_1, \dots, \mathcal{G}_{n_0-1}$  remains unchanged, because none of them contains  $b_{j_{n_0}}$ .

→[e] All other axioms come over unchanged because they contain no occurrences of  $b_{j_{n_0}}$ .

→[f] Clearly, an instance of MP (respectively gen) in deduction (A) maps to a valid instance of MP (respectively gen) in (A').

Furthermore, the last line of (A') is the wf  $F_{n_0}(y)$ , so applying gen to this last line gives a deduction in  $S_{n_0-1}$  of  $(\forall y)F_{n_0}(y)$ ; and this last formula is provably equivalent to  $(\forall x_{i_{n_0}})F_{n_0}(x_{i_{n_0}})$  (Exercise 3). Thus there exists a deduction

$$\vdash_{S_{n_0-1}} (\forall x_{i_{n_0}})F_{n_0}(x_{i_{n_0}}); \tag{C}$$

and deductions (C) and (B) together would make  $S_{n_0-1}$  inconsistent. Thus,  $n_0$  is not minimal. ■(claim)

**Exercise 3.** Use Proposition 4.18 (class notes March 23) to show that  $(\forall y)F_{n_0}(y)$  is provably equivalent to  $(\forall x_{i_{n_0}})F_{n_0}(x_{i_{n_0}})$ .

**Step 3.** Let  $S_\infty := \bigcup_{n=0}^{\infty} \text{Thm}(S_n)$ . Then  $S_\infty$  is consistent (Exercise 4). Let  $J$  be a consistent complete extension of  $S_\infty$ .

**Exercise 4.** Show that  $S_\infty$  is consistent.

**Step 4. The Model.**

The interpretation  $I$  defined below will turn out to be the promised model.

→ Let  $D = D_I$  be the set of all closed (variable-free) terms of the extended system.

→ Let  $\bar{a}_i = a_i$  and  $\bar{b}_i = b_i$  for all of the individual constants.

→ The definitions of  $\{\bar{f}_i^n\}$  are automatic: each function letter  $f_j^n$  actually already generates a genuine function  $\bar{f}_i^n: D^n \rightarrow D$ , namely

$$(t_1, \dots, t_n) \xrightarrow{\bar{f}_i^n} f_i^n(t_1, \dots, t_n),$$

where  $t_1, \dots, t_n$  are any closed terms.

→ For each predicate letter  $A_j^n$ , let  $\bar{A}_j^n$  be the  $n$ -ary predicate

$$\bar{A}_j^n(t_1, \dots, t_n) \text{ is true} \iff \vdash_J A(t_1, \dots, t_n). \quad (*)$$

**Exercise 5.** Let  $w$  be any valuation in this interpretation, and let  $t$  be any closed term in the extended system. Show that  $w(t) = t$ .

**Step 5.** I claim that for any closed wf  $\mathcal{A}$  of the extended system,

$$\vdash_J \mathcal{A} \iff I \models \mathcal{A}. \quad (\dagger)$$

**Proof of claim.** by induction on the number  $n$  of logical symbols in  $\mathcal{A}$ .

**Basis** ( $n = 0$ ) is exactly  $(*)$  above.

**Induction.** Let  $\mathcal{A}$  be closed and have  $n > 0$  logical symbols, and assume  $(\dagger)$  for any closed wf with  $\leq (n-1)$  logical symbols. There are several cases.

**Cases 1 and 2:**  $\mathcal{A}$  is  $\sim \mathcal{B}$ , or  $\mathcal{A}$  is  $\mathcal{B} \rightarrow \mathcal{C}$ . I will leave these cases as exercises.

**Exercise 6.** Complete the induction in the case  $\mathcal{A}$  is  $\sim \mathcal{B}$ .

**Exercise 7.** Complete the induction in the case  $\mathcal{A}$  is  $\mathcal{B} \rightarrow \mathcal{C}$ .

**Case 3:**  $\mathcal{A}$  is  $(\forall x)\mathcal{B}$  for some variable  $x$ , and  $\mathcal{B}$  is closed. Because  $\mathcal{B}$  is closed, I can apply the inductive hypothesis to  $\mathcal{B}$ :

$$\begin{aligned} & I \models \mathcal{A} \\ (\text{Prop. 3.27, March 2 } \rightarrow) & \iff I \models \mathcal{B} \\ (\text{induction, applied to } \mathcal{B} \rightarrow) & \iff \vdash_J \mathcal{B} \\ (\text{Prop 4.18 } \frac{1}{2}, \text{ March 2 } \rightarrow) & \iff \vdash_J \mathcal{A}. \end{aligned}$$

**Case 4:**  $\mathcal{A}$  is  $(\forall x)\mathcal{B}$  for some variable  $x$ , and  $\mathcal{B}$  is not closed. In this case, since  $\mathcal{A}$  is closed,  $\mathcal{B}$  is  $F_k(x_{i_k})$  for some  $k$ , and  $\mathcal{A}$  is  $(\forall x_{i_k})F_k(x_{i_k})$ .

I will first prove that if  $I \models \mathcal{A}$ , then  $\vdash_J \mathcal{A}$ . Suppose that

$$I \models \mathcal{A}. \quad (\alpha)$$

Because  $(\forall x_{i_k})F_k(x_{i_k}) \rightarrow F_k(b_{j_k})$  is also true in  $I$  (why?), so from  $(\alpha)$  and Prop. 3.26 (March 15), I get

$$I \models F_k(b_{j_k}).$$

I can now apply the inductive hypothesis to  $F_k(b_{j_k})$ , which has one less logical symbol than  $\mathcal{A}$ :

$$\vdash_J F_k(b_{j_k}). \quad (\gamma)$$

By appending two lines to this deduction, I can then get  $\vdash_J \mathcal{A}$ :

1.  $F_k(b_{j_k})$  ( $\gamma$ )
2.  $F_k(b_{j_k}) \rightarrow (\forall x_{i_k})F_k(x_{i_k})$   $\mathcal{G}_k \in S_\infty$
3.  $(\forall x_{i_k})F_k(x_{i_k})$  MP(1,2)

Next (and finally for this step), I will prove that if  $\vdash_J \mathcal{A}$ , then  $I \models \mathcal{A}$ . Suppose (to get a contradiction) that

$$\vdash_J \mathcal{A} \tag{\zeta}$$

but

$$\text{not } I \models \mathcal{A}. \tag{\eta}$$

By ( $\eta$ ), there is a valuation  $v_0$  in  $I$  for which  $f_{v_0}((\forall x_{i_k})F_k(x_{i_k})) = F$ , so that there is also a valuation  $w_0$  in  $I$  such that  $w_0 \stackrel{\sim}{i_k} v_0$  and such that

$$f_{w_0}(F_k(x_{i_k})) = F.$$

Now consider  $w_0$ .  $w_0(x_{i_k})$  is some element of  $D$ —that is, some closed term  $t$ . By Exercise 5, I also have that  $w_0(t) = t$ ; that is

$$w_0(x_{i_k}) = t = w_0(t).$$

This last says that  $w_0 = \text{sub}(w_0, t \rightarrow x_{i_k})$ , so I can apply the Substitution Theorem (with the same valuation  $w_0$  on both sides):

$$f_{w_0}(F_k(t)) = f_{w_0}(F_k(x_{i_k})) = F.$$

I thus have one valuation that does not satisfy  $F_k(t)$ , so that

$$\text{not } I \models F_k(t).$$

But  $F_k(t)$  is closed and contains one fewer logical symbols than does  $\mathcal{A}$ , so by induction we know that

$$\text{not } \vdash_J F_k(t). \tag{\theta}$$

On the other hand: I can append the following to deduction ( $\zeta$ ):

1.  $(\forall x_{i_k})F_k(x_{i_k})$  ( $\zeta$ )
2.  $(\forall x_{i_k})F_k(x_{i_k}) \rightarrow F_k(t)$   $\mathcal{K}5$
3.  $F_k(t)$  MP(1,2)

This gives

$$\vdash_J F_k(t),$$

which obviously contradicts ( $\theta$ ). This completes the induction. **■(claim)**

**Exercise 8.** How do I know that line 2. above is a valid instance of  $\mathcal{K}5$ ?

**Step 6.** Finally, I **claim** that for any wf  $\mathcal{A}$  of the extended system, **closed or not**,

$$\vdash_J \mathcal{A} \iff I \models \mathcal{A}.$$

**Proof of claim.** Let  $\mathcal{A}'$  be the universal closure of  $\mathcal{A}$ . Then

$$\begin{aligned} & I \models \mathcal{A} \\ \text{(Cor. 3.28, March 2 } \rightarrow) & \iff I \models \mathcal{A}' \\ \text{(Step 5 } \rightarrow) & \iff \vdash_J \mathcal{A}' \\ \text{(Prop 4.19, March 23 } \rightarrow) & \iff \vdash_J \mathcal{A}. \text{ **■(claim and theorem)**} \end{aligned}$$

**Corollary 1 (Proposition 4.47: the Löwenheim–Skolem Theorem).** *If  $S$  is a consistent first-order system, then  $S$  has a **DENUMERABLE** model!*

*Proof.* Look at Step 4 of the proof of the theorem: the set  $D$  is denumerable! **■**

**Corollary 2 (Proposition 4.41: the Adequacy Theorem for  $\mathcal{K}$ ).** *If  $\mathcal{A}$  is a logically valid wf, then*  
 $\vdash_{\mathcal{K}} \mathcal{A}$ .

*Proof.* I will prove the contrapositive. Suppose that not

$$\text{not } \vdash_{\mathcal{K}} \mathcal{A}.$$

Then also

$$\text{not } \vdash_{\mathcal{K}} \mathcal{A}',$$

where  $\mathcal{A}'$  is the universal closure of  $\mathcal{A}$ . But then by Proposition 4.37,  $\mathcal{K} \oplus [\sim \mathcal{A}']$  is a consistent first-order system, and so by Proposition 4.40, it has a model  $I$ . In this model,  $I \models \sim \mathcal{A}'$ , so  $\sim \mathcal{A}'$  is true and  $\mathcal{A}'$  is false. Thus, for at least one valuation  $v$  in this model,  $f_v(\mathcal{A}) = F$ . ■