A Proof of $\Longleftarrow$ Direction of Theorem 1.5 (§1.9, p.59)

Theorem. *Let $z = M(t, y)$ and $z = N(t, y)$ have continuous partial derivatives, and suppose, for all (x, y) in some rectangle R, that $\dfrac{\partial M}{\partial y}(t, y) = \dfrac{\partial N}{\partial t}(t, y) = K(t, y)$. Then there exists a function $z = \phi(t, y)$, defined on R, such that $\dfrac{\partial \phi}{\partial t} = M$ and $\dfrac{\partial \phi}{\partial y} = N$.*

Proof. Let (a, c) be the lower left-hand corner of R. For each (x, y) in R, put

$$\phi(t, y) := \int_a^t \int_c^y K(\widehat{t}, \widehat{y})\, d\widehat{y}\, d\widehat{t} + \int_a^t M(\widehat{t}, c)\, d\widehat{t} + \int_c^y N(a, \widehat{y})\, d\widehat{y}.$$

We have, for each $(t, y) \in R$,

$$\begin{aligned}
\frac{\partial \phi}{\partial t}(t, y) &= \frac{\partial}{\partial t}\left[\int_a^t \int_c^y K(\widehat{t}, \widehat{y})\, d\widehat{y}\, d\widehat{t}\right] + \frac{\partial}{\partial t}\left[\int_a^t M(\widehat{t}, c)\, d\widehat{t}\right] + \frac{\partial}{\partial t}\left[\int_c^y N(a, \widehat{y})\, d\widehat{y}\right] \\
&= \int_c^y K(t, \widehat{y})\, d\widehat{y} \qquad + \qquad M(t, c) \qquad + \qquad 0 \\
\left(K = \frac{\partial M}{\partial y} \longrightarrow\right) &= \Big[M(t, y) - M(t, c)\Big] + M(t, c) \\
&= M(t, y).
\end{aligned}$$

By a completely symmetric computation, one can show that $\dfrac{\partial \phi}{\partial y}(t, y) = N(t, y)$. ∎

Exercise. Show that $\dfrac{\partial \phi}{\partial y}(t, y) = N(t, y)$.