

Partial Fractions and Integrating Rational Functions

The term *partial fractions* refers to an algebraic technique which is usually included in a presentation of methods for integrating functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials. It is a fact that all such functions (which are called *rational functions*) can be integrated, at least in principle. In practice, some of the techniques you need are quite laborious, and others may sometimes be impossible to apply. I am going to ask you to learn to apply these techniques only in a few special cases, which I will discuss in class. Here, I want to give you an idea of the big picture, which is easy to grasp in outline.

Step I. If the degree of $f(x)$ is larger than or equal to the degree of $g(x)$, you can divide $f(x)$ by $g(x)$, getting quotient $q(x)$ and remainder $r(x)$, where $q(x)$ is a polynomial, and $r(x)$ is either zero or a polynomial whose degree less than the degree of $g(x)$. This will allow you to rewrite the integrand as

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

Since $\int q(x) dx$ is trivial to compute, this reduces the original integration problem to that of finding $\int \frac{r(x)}{g(x)} dx$ (where the degree of the numerator is smaller than that of the denominator). In class, I will discuss this step in detail; I want you to know this part well.

Step II. Factor $g(x)$ as far down as possible, using real numbers (both rational and irrational) for coefficients. It can be proved that $g(x)$ can in principle be factored into factors of degree at most 2. For example, here is a typical complete factorization:

$$g(x) = 3(x - 3)^3(x - \pi)(x^2 + \sqrt{2}x + 4)^3.$$

Unfortunately, these factors are sometimes impossible to find in practice. I will bother you with only very small cases in class.

Step III. Using the factorization from Step II, it is algebraically possible to write $\frac{r(x)}{g(x)}$ as a sum of simpler fractions, using powers of the factors of $g(x)$ for denominators. Here is an example: if

$$g(x) = (x - 5)^3(x + 3)(x^2 + 2x + 3)^2,$$

then you can find (unique) constants so that

$$\frac{r(x)}{g(x)} = \frac{c_1}{x - 5} + \frac{c_2}{(x - 5)^2} + \frac{c_3}{(x - 5)^3} + \frac{c_4}{x + 3} + \frac{c_5 + d_1x}{x^2 + 2x + 3} + \frac{c_6 + d_2x}{(x^2 + 2x + 3)^2}. \quad (1)$$

This is called the *partial fraction decomposition* of $\frac{r(x)}{g(x)}$. Finding the constants is routine but laborious; I will discuss only very simple examples in class.

Step IV. All of the terms you get in (1) can in fact be integrated. In class, I will discuss how to integrate some of them but not all.