Root Test. Suppose that the following limit exists:

$$L = \lim_{k\to\infty} \sqrt[k]{|a_k|}.$$

Then

$$\begin{cases} \textbf{[a]:}\ L < 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \text{ converges absolutely;} \\ \textbf{[b]:}\ L > 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \text{ diverges;} \\ \textbf{[c]:}\ \text{if } L = 1, & \sum_{k=0}^{\infty} a_k \text{ might converge absolutely, converge conditionally, or diverge.} \end{cases}$$

Proof. **[a].** Assume $L < 1$, and choose any number r that is strictly between L and 1: that is, $L < r < 1$. Now $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L < r$, so for all k large enough—say, for all $k \geq K_0$—it must be true that

$$\sqrt[k]{|a_k|} \leq r,$$

which implies that

$$|a_k| \leq r^k.$$

Next: since $r < 1$, the geometric series

$$\sum_{k=K_0}^{\infty} r^k = r^{K_0} + r^{K_0} \cdot r + r^{K_0} \cdot r^2 + \cdots \text{ converges to } \frac{r^{K_0}}{1-r};$$

so by the PVCT, $\sum_{k=K_0}^{\infty} |a_k|$ also converges. Finally, this means that $\sum_{k=K_0}^{\infty} a_k$ converges absolutely.

[b]: If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L > 1$, then for all k large enough—say, for all $k \geq K_0$—it must be true that

$$\sqrt[k]{|a_k|} \geq 1,$$

which implies that

$$|a_k| \geq 1^k = 1.$$

This makes it impossible for $\lim_{k\to\infty} a_k$ to equal zero, so by the Test for Divergence, $\sum_{k=K_0}^{\infty} a_k$ diverges.

[c]: To prove the third assertion, it is sufficient to exhibit three series—one absolutely convergent, the second conditionally convergent, and the third divergent—for which $L = 1$. Three series that work are $\sum_{k=1}^{\infty} \frac{1}{k^2}$, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$, and $\sum_{k=1}^{\infty} \frac{1}{k}$. The first of these converges absolutely (p series, $p = 2$), the third one diverges (p series, $p = 1$), the second one converges conditionally (Alt. Series Test). Moreover, routine L'Hospital's Rule computations show that $L = 1$ for all of these series. ∎

Extra credit: Ten points extra credit on the quiz total of anyone who computes L for these three series.