

Dealing with Proofs.

A proof is an organized piece of reasoning, whose purpose is to establish the truth of a mathematical statement; hence, in mathematics, proof is the essential tool for advancing knowledge and for communicating it. In your intermediate and advanced math courses, you will need to be able to understand and absorb proofs, and you will frequently be called upon to construct proofs of your own. These tasks are daunting but by no means undoable, and I hope this handout will give you some practical help with them.

A. Some Tips on Reading Proofs.

1. Read proofs more than once. When you first read a difficult proof, your goal should be to understand its logical structure. You may not know how the writer of the proof thought of employing the steps (s)he used¹, and you may not see why the theorem is interesting or worth proving. Often, those understandings come only later, on a second or third reading. In the same way, you should not expect to understand every line of every proof discussed in class; but the ideas should become clear as you reread your class notes.

2. Write things down. If you really want to master the material—to make it yours—you should write notes meant for your eyes only. Keep a journal with your insights and questions, outlines of homework solutions you have found, and—perhaps—recopy your class notes. (Such notes will be of immense help, by the way, when you are studying for an exam.)

B. Some Tips on Writing Proofs.

Say you have successfully tackled a proof problem: you see why the assertion must be true. You now face the task of writing up your solution. Even if you have solved the problem completely and you understand your solution thoroughly, it is still not easy to write a clear, coherent account of your proof. This job demands thought, discipline, effort, and—usually—revisions.² The following tips and guidelines should make your efforts more effective and improve your final draft.

1. Make sure you have verified what you are writing. Work through all obstacles. Focus on and dissolve any problems, and strengthen any weak links in your chain of reasoning. You will find that clear understanding will actually solve many (although not all!) of the writing problems in advance and will cut down on the amount of revising you will need to do: knowing what you want to say will often lead you to a way of saying it.

2. If your solution is not complete, write what you have and explain how it falls short. Suppose that you can show part but not all of what you were asked to, or that you need to add an assumption to make your proof go through. *Include acknowledgements of any such shortcomings.* Doing so improves your writeup: it tells the reader that you are aware of the gaps—that you yourself know when you have not proved something.

3. Have an outline of your entire proof in mind as you write. Keep track of what you are assuming, what you are proving, and where you are in your proof. You should be especially careful if your proof is an argument by contradiction, or if you are arguing the *contrapositive*—that is, if you are arguing from the negation of the conclusion to the negation of the hypothesis.

¹ When a proof employs truly original ideas, you may never know how the prover ever thought of them. Nonetheless, now that the ideas have been written down, you can make them yours. In mathematics, more than in any other field, the ideas of the giants can be absorbed, understood, and used by the rest of us.

² Part of your job in this course will be to hone and strengthen your proof-writing skills, and part of my job will be to assist you. For the most part, this assistance will come from worked solutions to assigned problems, from written comments on your submitted solutions, and from face-to-face discussion.

4. Include enough words... Your attempt to communicate your ideas to the reader will often stand or fall by how much “connective tissue” you include. I am talking about sentences like these:

“The problem as stated is equivalent to statement XXXX. I will prove XXXX.”

[If you plan to prove $X \implies Y$ by proving the contrapositive $\sim(Y) \implies \sim(X)$]: “Suppose, on the contrary, that Y is false.”

“I must show that XXXX and YYYY together imply ZZZZ. I will do this by showing how to get a contradiction from XXXX, YYYY, and not-ZZZZ.”

“Such a function cannot possibly exist: assuming that you have any such function leads to the following contradiction.”

Such sentences serve as logical guide-posts for the reader: they set out the large-scale structure of the argument—the main ideas that made your proof work.

5. ...but don’t include extra words. Don’t bury your argument in irrelevant details. In the process of finding a proof, you may start by working specific examples to get initial insights into what is going on; you may first prove a special case, using extra assumptions; or you may discover side facts that you don’t wind up using in the final argument. Exclude them from the final draft. Such irrelevancies are like the scaffolding that used to construct a building: after the building is finished, the scaffolding is no longer needed and hides the completed structure.

6. Be clear and precise. Imprecise mathematical prose often indicates lingering fuzziness in the mathematics itself, and you will find that clarifying your explanation will often clarify your thinking. You can make your prose clearer by writing with the following goals in mind. (I’m sure there are others I haven’t thought of!) Define any new terms that you introduce. When you introduce a variable, state precisely what it represents. Make sure that each pronoun you use—each “it,” “they” or “them”—has an unambiguous referent. Be careful in your use of logic: make all quantifiers (“for some” and “for all” phrases) explicit; don’t confuse an implication with its converse; be especially careful if you must express the negation of a complicated logical statement.

7. Choose your notation carefully. Well-chosen notation streamlines exposition and sometimes actually simplifies an argument, making it easier to follow.² On the other hand, poorly chosen or imprecise notation makes it hard for the reader to follow your argument and may wind up confusing even you.

8. Read your proof aloud. After you have written your proof down, read it aloud. This will help you gauge how it will sound (and look) to others. You may find an error, a gap, or—even if your solution is correct—a place where your explanation is unclear.

² For example, here is a puzzle: are my grandmothers’ great-grandmothers the same people as my great-grandmothers’ grandmothers? Good notation is of immense help for this one.