A Property of Polynomials

I. Introduction.

It is not hard to prove that for any real number a, a polynomial of degree n can be expanded in powers of $(x-a)$:

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = \sum_{k=0}^{n} c_k(x-a)^k. \tag{1}$$

One can prove this, for example, by finding and solving a system of linear equations satisfied by the coefficients $(c_0, c_1, \ldots, c_n)$. However, there turns out to be a much faster way to find the values of these coefficients, and this faster method can be adapted to the problem of finding a power series—think of it as an "infinite polynomial"—that will equal a non-polynomial function, such as $y = \sin(x)$ or $y = e^x$. This handout will explain this method.

II. An Example.

Let's begin with an example, which I will construct by working backwards. Consider the polynomial

$$p(x) = \frac{1}{3}(x-1)^3 + 3(x-1)^2 + 2(x-1) + \frac{10}{3}. \tag{2}$$

By multiplying this out, you can check that when written in the usual way,

$$p(x) = \frac{1}{3}x^3 + 2x^2 - 3x + 4. \tag{3}$$

Now, suppose we start with equation (3), and we want to find equation (2). I claim that you can find the coefficients by filling in the following chart, which we will fill out in class together. (In the second column, "$p^{(k)}(x)$" means the k^{th} derivative of the polynomial $y = p(x)$; the 0^{th} derivative of any function is understood to mean the function itself.)

k	$p^{(k)}(x)$	$p^{(k)}(1)$	$\frac{p^{(k)}(1)}{k!}$
0			
1			
2			
3			

As you see, the last column has recovered the coefficients that must be put next to the powers of $(x-a)$. This is no coincidence; this works for any polynomial, as you will see.

III. The General Case.

We will construct a chart of this kind for an arbitrary polynomial, expanded in powers of $(x-a)$ for an arbitrary a. We will keep careful track of what happens as we fill it in.

k	$p^{(k)}(x)$	$p^{(k)}(a)$	$\frac{p^{(k)}(a)}{k!}$
0	$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_k(x-a)^k + \cdots + c_n(x-a)^n$	c_0	c_0
1	$c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + kc_k(x-a)^{k-1} + \cdots + nc_n(x-a)^{n-1}$	c_1	c_1
2	$2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 \cdots + k(k-1)c_k(x-a)^{k-2} + \cdots + n(n-1)c_n(x-a)^{n-2}$	$2c_2$	c_2
3	$6c_3 + 24c_4(x-a) + \cdots + k(k-1)(k-2)c_k(x-a)^{k-3} + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$	$6c_3$	c_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$k!\,c_k + \cdots + n(n-1)\cdots(n-k+1)c_n(x-a)^{n-k}$	$k!\,c_k$	c_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$n!\,c_n$	$n!\,c_n$	c_n

In short: evaluating k^{th} derivative of $p(x)$ at $x=a$ gives as output the constant of the k^{th} derivative, which turns out to be $k!\,c_k$. Thus, for a polynomial $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots c_k(x-a)^k + \cdots + c_n(x-a)^n$, we have

$$p^{(k)}(a) = k!\,c_k,$$

so that

$$c_k = \frac{p^{(k)}(a)}{k!}.$$

IV. An Application: The Binomial Theorem.

As mentioned above, the principal application of this property is to the construction of power series that equal non-polynomial functions; this topic will be discussed in class not in this handout. However, one can also use this property to prove the Binomial Theorem, which tells you how to multiply out $(A+B)^n$. To explain the theorem, I need to introduce the so-called "binomial coefficients" $\binom{n}{k}$, defined for integers $0 \le k \le n$. The definition is

$$\binom{n}{0} := 1; \text{ and, for } 1 \le k \le n, \binom{n}{k} := \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Note that for $1 \le k \le n$, the numerator is the product of the highest k numbers, while the denominator is the product of the lowest k numbers; for example,

$$\binom{4}{2} = \frac{4 \times 3}{2 \times 1} = 6; \text{ and } \binom{13}{4} = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715.$$

As it turns out, you will also find these numbers in *Pascal's Triangle*, rows 0 through 4 of which are shown below:

$$\begin{tikzcd}
 & & & & 1 \arrow[dl] \arrow[dr] & & & & \\
 & & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & & \\
 & & 1 \arrow[dl] \arrow[dr] & & 2 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & \\
 & 1 \arrow[dl] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & \\
1 & & 4 & & 6 & & 4 & & 1
\end{tikzcd}$$

This triangle is constructed by putting 1's down the sides and then filling in each middle position by adding the numbers in the two positions above it. When you do that, the n^{th} row turns out to contain the numbers $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots$; for example, the middle number in the bottom row is $\binom{4}{2}$, which was computed above. I can now state the Binomial Theorem.

Binomial Theorem.

$$(A+B)^n = B^n + \binom{n}{1}AB^{n-1} + \binom{n}{2}A^2B^{n-2} + \cdots + \binom{n}{n-1}A^{n-1}B + A^n \left(= \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \right). \tag{4}$$

The proof follows easily from the special case below; see Exercise (5). ∎

Binomial Theorem, Special Case. *For any* $n \ge 0$,

$$(x+1)^n = \sum_{k=1}^{n} \binom{n}{k} x^k. \tag{5}$$

(Observe that you get the special case from the general case by putting x in for A and 1 in for B.)

Proof. Make a chart for the polynomial $p(x) = (x+1)^n$, taking $a = 0$:

k	$p^{(k)}(x)$	$p^{(k)}(0)$	$\frac{p^{(k)}(0)}{k!}$
0	$(x+1)^n$	1	1
1	$n(x+1)^{n-1}$	n	n
2	$n(n-1)(x+1)^{n-2}$	$n(n-1)$	$\frac{n(n-1)}{2!}$
3	$n(n-1)(n-2)(x+1)^{n-3}$	$n(n-1)(n-2)$	$\frac{n(n-1)(n-2)}{3!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The numbers in the rightmost column are exactly the binomial coefficients $\binom{n}{k}$. Therefore, when you expand $(1+x)^n$ in powers of x, the result is equation (5). ∎

V. Exercises.

Exercise 1. Use the Binomial Theorem (equation (4)) to multiply out $(x+2y)^4$.

Exercise 2. Expand $p(x) = 3 + 5x - 2x^2 + 7x^3$ in powers of $(x-2)$.

Exercise 3. Expand $p(x) = x^5$ in powers of $(x+2)$.

Exercise 4. If you were to expand $p(x) = (x+2)^{100}$ in powers of $(x-5)$, what would the coefficient of $(x-5)^3$ be?

Exercise 5. Derive equation (4) from equation (5). *Hint*: For $B = 0$, explain why both sides of equation (4) equal A^n. Then for $B \neq 0$, start by showing that

$$(A+B)^n = B^n \left(\frac{A}{B} + 1\right)^n .$$