A Fundamental Theorem on Power Series

The theorem below guarantees that for any power series $\sum c_n x^n$, one of three things must happen:

[a]: The series converges only for $x = 0$;
[b]: The series converges for all x; or
[c]: The set of x's for which the series converges is an interval of the form $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$.

Theorem. *Let $x \neq 0$, and say that $\sum_{k=0}^{\infty} c_k x^k$ converges (either absolutely or conditionally). If y is any number such that $-|x| < y < |x|$, then $\sum_{k=0}^{\infty} c_k y^k$ converges absolutely.*

Proof. Since $\sum_{k=0}^{\infty} c_k x^k$ converges, $\lim_{k\to\infty} c_k x^k = 0$ (by Test for Divergence). From this it follows that all but finitely many terms have absolute value ≤ 1, which means that there is some number K such that, for all n,

$$|c_n x^n| \leq K. \tag{$*$}$$

Now choose any number y such that $-|x| < y < |x|$. Since $|y| < |x|$, we have

$$\frac{|y|}{|x|} = \left|\frac{y}{x}\right| < 1.$$

Call this number r:

$$\frac{|y|}{|x|} = \left|\frac{y}{x}\right| = r < 1.$$

We can now apply the PVCT: for any n,

$$\begin{aligned} & 0 \leq |c_n y^n| \\ (\text{multiply and divide by } x^n \longrightarrow) &= \left| c_n x^n \frac{y^n}{x^n} \right| \\ &= |c_n x^n| \left|\frac{y}{x}\right|^n \\ &= |c_n x^n| r^n \\ (\text{by } (*) \longrightarrow) &\leq K r^n, \end{aligned}$$

and $\sum K r^n$ is a convergent geometric series; so by PVCT, $\sum |c_n y^n|$ converges—that is, $\sum c_n y^n$ converges absolutely. ∎