The Propositional Content of Predicate-Calculus *wf*s; Predicate-Calculus Tautologies

In this handout, I will construct a bijection $\mathcal{T}$ from the set of $\pounds$–*wf*s onto the set of L–*wf*s, the purpose of which is to isolate the L–content of each $\pounds$–*wf*. I will then use $\mathcal{T}$ to identify the $\pounds$–tautologies. My construction seems to me to be of exactly the same flavor as are others one encounters in this subject: completely precise but perhaps not completely transparent.

Step 1. L-complexity. Before defining $\mathcal{T}$, I need to construct an integer-valued function on the $\pounds$–*wf*s that will measure their propositional complexity, by counting up the number of "L–visible" occurrences of $\sim$ and $\to$. I will first isolate the $\pounds$–*wf*s of propositional complexity zero; these are the $\pounds$–*wf*s in which the Propositional Calculus can discern no logical structure whatsoever.

> **Definition 1.** Say that $\pounds$–*wf* $\mathcal{A}$ is *L–irreducible* iff it is either atomic or of the form $(\forall x_i)\mathcal{B}$ for some $\pounds$–*wf* $\mathcal{B}$.

I can now construct the L–complexity function d that I will need.

> **Definition 2.** Define the *L–complexity* d of $\pounds$–*wf* A as follows:
> $\to$: If $\mathcal{A}$ is L–irreducible, then $d(\mathcal{A}) := 0$;
> $\to$: If $\mathcal{A}$ is of the form $(\sim\mathcal{B})$, then $d(\mathcal{A}) := d(\mathcal{B}) + 1$;
> $\to$: If $\mathcal{A}$ is of the form $(\mathcal{B} \to \mathcal{C})$, then $d(\mathcal{A}) := d(\mathcal{B}) + d(\mathcal{C}) + 1$.

Step 2. The bijection $\mathcal{T}$. I will define a function $\mathcal{T}(\mathcal{A}) = \hat{\mathcal{A}}$ (from $\pounds$–*wf*s to L–*wf*s) which will be designed to isolate the L–content of each $\pounds$–*wf* $\mathcal{A}$. $\mathcal{T}$ will be defined by recursion over $d(\mathcal{A})$.

$\to d = 0$:

Choose any enumeration $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots)$ of the the L-irreducible *wf*s and put

$$\mathcal{T}(\mathcal{A}_i) := p_i, \;\; i = 1, 2, 3, \ldots.$$

$\to d > 0$:

(α), if $\mathcal{A}$ is of the form $(\sim\mathcal{B})$, then put $\mathcal{T}(\mathcal{A}) := \big(\sim\mathcal{T}(\mathcal{B})\big)$;

(β), if $\mathcal{A}$ is of the form $(\mathcal{B} \to \mathcal{C})$, then put $\mathcal{T}(\mathcal{A}) := \big(\mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{C})\big)$.

Exercise 1. Show that $\mathcal{T}$ is indeed a bijection from the set of $\pounds$–*wf*s onto the set of L–*wf*s.

Step 3. L–valuations from $\pounds$–valuations. Let v be any $\pounds$–valuation in any interpretation of $\pounds$. Associate to v the L–valuation $\tilde{v}$ obtained by putting, for each L–*wf* $\hat{\mathcal{A}}$,

$$\tilde{v}(\hat{\mathcal{A}}) := f_v\left(\mathcal{T}^{-1}(\hat{\mathcal{A}})\right).$$

Exercise 2. Show that $\tilde{v}$ is a a propositional-calculus valuation. (That is, show that it satisfies the definition.)

> **Definition 3.** Say that $\pounds$-wf $\mathcal{A}$ is a *tautology in $\pounds$* iff $\mathcal{T}(\mathcal{A})$ is a tautology in L.

Exercise 3. Show that every tautology in $\pounds$ is logically valid.