The Completeness of the Propositional Calculus

Hilbert's program for the Propositional Calculus met with complete success: the "Propositional Calculus Robot" functions perfectly. When taken together with the proposition from class that nontautologies are not deducible, the extra problem in Assignment 4 shows[1] that L is consistent. The purpose of this handout is to show that L is also *complete*. (This means roughly, that L can deduce every *wf* we'd like it to be able to. Below, I will make the notion precise, in two different ways.) The main result will be established in three lemmas; but before getting to those, I need to establish a few technical details about L.

I. Five Formal Deductions and a Deduction Rule.

1: $\vdash_L \mathcal{B} \to \sim\sim\mathcal{B}$.

1.	$\sim\sim\sim\mathcal{B} \to \sim\mathcal{B}$	Deduction Theorem applied to instance of Hamilton #2b (p 36)
2.	$(\sim\sim\sim\mathcal{B} \to \sim\mathcal{B}) \to (\mathcal{B} \to \sim\sim\mathcal{B})$	$L3$
3.	$\mathcal{B} \to \sim\sim\mathcal{B}$	MP(1,2)

2: $\vdash_L (\mathcal{A} \to \mathcal{B}) \to (\sim\mathcal{B} \to \sim\mathcal{A})$.

This is Hamilton #3b (p 36).

3: $\vdash_L (\mathcal{A} \to \sim\mathcal{A}) \to \sim\mathcal{A}$.

1.	$(\sim\sim\mathcal{A} \to \sim\mathcal{A}) \to \sim\mathcal{A}$	Instance of Hamilton, Prop. 2.11b (p 35)
2.	$(\mathcal{A} \to \sim\mathcal{A}) \to (\sim\sim\mathcal{A} \to \sim\mathcal{A})$	Instance of #**2** above
3.	$(\mathcal{A} \to \sim\mathcal{A}) \to \sim\mathcal{A}$	HS(2,1)

4: $\vdash_L \mathcal{A} \to \big((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}\big)$.

This can be obtained by applying the Deduction Theorem twice to the obvious three-line deduction of $[\mathcal{A}, \mathcal{A} \to \mathcal{B}] \vdash_L \mathcal{B}$.

5: $[\mathcal{A}, \sim\mathcal{B}] \vdash_L \sim(\mathcal{A} \to \mathcal{B})$.

1.	$\mathcal{A} \to \big((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}\big)$	#**4** above
2.	$\mathcal{A}$	Assumption
3.	$(\mathcal{A} \to \mathcal{B}) \to \mathcal{B}$	MP(2,1)
4.	$\big((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}\big) \to \big(\sim\mathcal{B} \to \sim(\mathcal{A} \to \mathcal{B})\big)$	Instance of #**2** above
5.	$\sim\mathcal{B} \to \sim(\mathcal{A} \to \mathcal{B}))$	MP(3,4)
6.	$\sim\mathcal{B}$	Assumption
7.	$\sim(\mathcal{A} \to \mathcal{B})$	MP(6,5)

Deduction Rule. If

$$\begin{cases} (\alpha)\ \Gamma \oplus [\mathcal{A}] \vdash_L \mathcal{C} \quad \text{and} \\ (\beta)\ \Gamma \oplus [\mathcal{B}] \vdash_L \mathcal{C}, \end{cases}$$

then

$$\Gamma \oplus [\mathcal{A} \vee \mathcal{B}] \vdash_L \mathcal{C}.$$

(Recall that "$\mathcal{A} \vee \mathcal{B}$" denotes $\sim\mathcal{A} \to \mathcal{B}$.)

1.	$\mathcal{A} \to \mathcal{C}$	Deduction Theorem applied to (α)
2.	$\mathcal{B} \to \mathcal{C}$	Deduction Theorem applied to (β)
3.	$\sim\mathcal{A} \to \mathcal{B}$	Assumption
4.	$\sim\mathcal{A} \to \mathcal{C}$	HS(3,2)
5.	$(\mathcal{A} \to \mathcal{C}) \to (\sim\mathcal{C} \to \sim\mathcal{A})$	Instance of #**2** above
6.	$\sim\mathcal{C} \to \sim\mathcal{A}$	MP(1,5)
7.	$\sim\mathcal{C} \to \mathcal{C}$	HS(6,4)
8.	$(\sim\mathcal{C} \to \mathcal{C}) \to \mathcal{C}$	Instance of Hamilton, Prop. 2.11b (p35)
9.	$\mathcal{C}$	MP(7,8)

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[1] The methods used in class, for the homework problem, and in this handout all pass Intuitionistic muster.

II. The Main Lemmas.

Proposition C1. *Let $\mathcal{A}$ be a proposition in distinct proposition letters $\{p_1, \dots p_m\}$, and focus on any one line (line j, say) of the 2^m lines of the truth table of $\mathcal{A}$. Using this line j, set*

$$q_i := \begin{cases} p_i, & \text{if } p_i \text{ is true in line } j\text{; and} \\ \sim p_i, & \text{if } p_i \text{ is false in line } j. \end{cases}$$

Then: **[a]**, *if $\mathcal{A}$ is true in line j, $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{A}$; and* **[b]**, *if $\mathcal{A}$ is false in line j, $[q_1, \dots, q_m] \underset{L}{\vdash} \sim\mathcal{A}$.*

Proof by induction by the number n of occurrences of "$\sim$" and "$\rightarrow$" in $\mathcal{A}$.

Basis $(n = 0)$. Then $\mathcal{A}$ is p_i. If p_i is true in line j, then q_i is p_i, so $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{A}$ by the one-line deduction

1. p_i assumption

On the other hand, if p_i is false in line j, then q_i is $\sim p_i$, so $[q_1, \dots, q_m] \underset{L}{\vdash} \sim\mathcal{A}$ by the one-line deduction

1. $\sim p_i$ assumption

Inductive step. Assume both **[a]** and **[b]** hold for all propositions in $\{p_1, \dots, p_m\}$ with $0 \le k \le (n-1)$ occurrences of "$\sim$" and "$\rightarrow$", and let $\mathcal{A}$ have n occurrences of "$\sim$" and "$\rightarrow$".

Case I: $\mathcal{A}$ is $\sim\mathcal{B}$ for some proposition $\mathcal{B}$.

Case I(α): If $\mathcal{A}$ is true (so $\mathcal{B}$ is false), then by induction $[q_1, \dots, q_m] \underset{L}{\vdash} \sim\mathcal{B}$; that is, $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{A}$.

Case I(β): If $\mathcal{A}$ is false (so $\mathcal{B}$ is true), then by induction $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{B}$. Add to this deduction:

ℓ. $\mathcal{B}$
$\ell+1$. $\mathcal{B} \rightarrow \sim\sim\mathcal{B}$ Formal deduction #**1** of section **I**
$\ell+2$. $\sim\sim\mathcal{B}$ MP(ℓ,$\ell+1$),

and this is what we want, because $\sim\sim\mathcal{B}$ is $\sim\mathcal{A}$.

Case II: $\mathcal{A}$ is $\mathcal{B} \rightarrow \mathcal{C}$ for some propositions $\mathcal{B}$ and $\mathcal{C}$.

Case II(α): Say $\mathcal{B}$ is false (so $\mathcal{A}$ is true). By induction, we have $[q_1, \dots, q_m] \underset{L}{\vdash} \sim\mathcal{B}$. Add to this deduction:

ℓ. $\sim\mathcal{B}$
$\ell+1$. $\sim \mathcal{B} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$ Hamilton, Prop. 2.11a(*p* 35)
$\ell+2$. $\mathcal{B} \rightarrow \mathcal{C}$ MP(ℓ,$\ell+1$)

Case II(β): Say $\mathcal{C}$ is true (so $\mathcal{A}$ is true). By induction, we have $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{C}$. Add to this deduction:

ℓ. $\mathcal{C}$
$\ell+1$. $\mathcal{C} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$ $L1$
$\ell+2$. $\mathcal{B} \rightarrow \mathcal{C}$ MP(ℓ,$\ell+1$)

Case II(γ): Say $\mathcal{B}$ is true and $\mathcal{C}$ is false (so $\mathcal{A}$ is false). By induction, we have both $[q_1, \dots, q_m] \underset{L}{\vdash} \mathcal{B}$ and $[q_1, \dots, q_m] \underset{L}{\vdash} \sim\mathcal{C}$. Concatenate these deductions, and add the following:

k. $\mathcal{B}$
ℓ. $\sim\mathcal{C}$
$\vdots$ $\vdots$ (Insert the lines of formal deduction #**5** of section**I**)
$\ell+7$. $\sim (\mathcal{B} \rightarrow \mathcal{C})$,

and this is what we want, because $\sim (\mathcal{B} \rightarrow \mathcal{C})$ is $\sim\mathcal{A}$. (Induction is complete.) ∎

Proposition C2. *Again, let $\mathcal{A}$ be a proposition in distinct proposition letters $\{p_1, \dots, p_m\}$. If $\mathcal{A}$ is a tautology, then $[p_1 \vee (\sim p_1), p_2 \vee (\sim p_2), \dots, p_m \vee (\sim p_m)] \underset{L}{\vdash} \mathcal{A}$.*

Proof is by $(2^{m-1} + 2^{m-2} + 2^{m-3} + \cdots + 2 + 1)$ applications of the deduction rule established in **I** above. (Let me call this "Deduction Rule $(*)$.") The argument, as you will see, has the same structure as the chart for an elimination tournament.

First: Because $\mathcal{A}$ is a tautology, Proposition C1 above gives 2^m deductions of $\mathcal{A}$ from 2^m different sequences of assumptions, namely:

(1) $[p_1, \ldots, p_{m-1}, p_m] \vdash_L \mathcal{A}$
(2) $[p_1, \ldots, p_{m-1}, \sim p_m] \vdash_L \mathcal{A}$
(3) $[p_1, \ldots, \sim p_{m-1}, p_m] \vdash_L \mathcal{A}$
(4) $[p_1, \ldots, \sim p_{m-1}, \sim p_m] \vdash_L \mathcal{A}$
$\vdots$ $\vdots$
$(2^m))$ $[\sim p_1, \ldots, \sim p_{m-1}, \sim p_m] \vdash_L \mathcal{A}$

Now apply Rule $(*)$ to each of the 2^{m-1} adjacent pairs of lines above, using in each case the assumptions p_m and $\sim p_m$:

$(1')$ $[p_1, \ldots, p_{m-1}, p_m \vee \sim p_m] \vdash_L \mathcal{A}$
$(2')$ $[p_1, \ldots, \sim p_{m-1}, p_m \vee \sim p_m] \vdash_L \mathcal{A}$
$\vdots$ $\vdots$
$(2^{(m-1)\prime})$ $[\sim p_1, \ldots, \sim p_{m-1}, p_m \vee \sim p_m] \vdash_L \mathcal{A}$

Now apply Rule $(*)$ to each of the 2^{m-2} adjacent pair of lines above, using in each case the assumptions the p_{m-1} and $\sim p_{m-1}$:

$(1'')$ $[p_1, \ldots, p_{m-1} \vee \sim p_{m-1}, p_m \vee \sim p_m] \vdash_L \mathcal{A}$
$\vdots$ $\vdots$
$(2^{(m-2)\prime\prime})$ $[\sim p_1, \ldots, p_{m-1} \vee \sim p_{m-1}, p_m \vee \sim p_m] \vdash_L \mathcal{A}$

Continuing in this way, after a total of $(2^{m-1} + 2^{m-2} + 2^{m-3} + \cdots + 2 + 1)$ applications of $(*)$, you finally arrive at one single deduction

$$[p_1 \vee (\sim p_1), p_2 \vee (\sim p_2), \ldots, p_m \vee (\sim p_m)] \vdash_L \mathcal{A}. \; ∎ \tag{†}$$

Proposition C3. *Let $\mathcal{A}$ be a proposition in distinct proposition letters $\{p_1, \ldots p_m\}$. If $\mathcal{A}$ is a tautology, then $\vdash_L \mathcal{A}$.*

Proof. **Exercise 1.** ∎ (*Hint*: Tinker with the justification side of deduction (†).)

A tautology, note, is a proposition that expresses a logical truth—that is, one that must be true regardless of what constituent statements it comprises. The first criterion for completeness of L is for all such logical truths to be deducible in L, and Proposition C3 says that L is complete in this sense. Another possible precise definition of "completeness" is a negative one: it says that if you cannot add any axioms to L, without rendering the system inconsistent. As it turns out, L is **not** complete in this negative sense, as we will see very soon; but it does have the property that if you added a fourth axiom *schema* that was not already a theorem, then the new system could not be consistent.

Proposition C4. *Let $\mathcal{A}$ be a wf schema that is not a tautology, and let L^+ be the system L with $\mathcal{A}$ added as axiom schema L4. Then L^+ is not consistent.*

Proof. **Exercise 2.** ∎ (*Hint*: Start by finding an instance of $\mathcal{A}$ that is identically false.)