An Outline of Some Basic Theorems on Infinite Series

I. Introduction

In class, we will be discussing the fact that well-behaved functions can be expressed as "infinite sums" or "infinite polynomials." For example, I will show below that for $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k;$$

another important and striking example—you must take my word for this at the moment!—is that for all x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(where $k! = 1 \times 2 \times \cdots \times (k-1) \times k$). In order to understand these infinite sums—for example, in order to know what sorts of operations on them are legal under what conditions—mathematicians studied them carefully; this handout outlines some of what they learned. I mean this outline as a supplement to your text (Chapter 8, §§1–5) not as a replacement for it; my hope is that the handout will help you sort and organize the large amount of difficult material that you must now master. I have included a few proofs; for the most part, however, the theorems will be stated here without proof, since I will be discussing the proofs in class. Also, there is one place where the order of my outline is quite different from the order of topics as presented in the text: I prefer to discuss the Absolute Convergence Test (Theorem (S6)) before talking about the Root Test and the Ratio Test (Theorems (S7) and (S8)). This eliminates the need to introduce the Root and Ratio Tests for a second time.

II. The Fundamental Definition

Suppose you have an infinite sequence of numbers that you want to add up. The way to make sense of this is to add more and more of them and to see whether these *finite* sums are approaching a limit. More precisely:

Definition 1. *Let* $\{a_k\} = \{a_0, a_1, \ldots\}$ *be a sequence of numbers. The corresponding* **sequence of partial sums** *is*

$$S_n = a_0 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

Definition 2. *Suppose* $\lim_{n\to\infty} S_n$ *exists. (Call the limit* S*.) Then one says that the series* $\sum_{k=0}^{\infty} a_k$ **converges** *and that its sum equals* S*. If* $\lim_{n\to\infty} S_n$ *does not exist, then one says that* $\sum_{k=0}^{\infty} a_k$ **diverges** *(or does not exist).*

One can summarize these two definitions by saying:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k \right);$$

the series converges if and only if this limit exists. Thus the sum of a series is defined to be the limit of a sequence—the sequence of partial sums. This means that knowledge of which series converge and which ones diverge is based on understanding how limits of sequences behave. I will discuss this topic in the next section.

III. Limits of Sequences

I will treat the notion "$\lim_{n\to\infty} S_n = S$" as self-explanatory. Many of the theorems here are just like the corresponding theorems for limits of functions. Here is a list of the theorems I will need:

Theorem (L1). $\lim_{n\to\infty} C = C$. ∎

Theorem (L2). *If* $\lim_{n\to\infty} S_n = S$ $\lim_{n\to\infty} T_n = T$ *and* c *is any number, then* $\lim_{n\to\infty} c \cdot S_n = c \cdot S$; $\lim_{n\to\infty}(S_n \pm T_n) = S \pm T$; $\lim_{n\to\infty}(S_n \cdot T_n) = S \cdot T$; *and, if* $T \neq 0$, *then* $\lim_{n\to\infty}(S_n/T_n) = S/T$. ∎

Theorem (L3). *Let* $\{S_n\}$ *be either a nondecreasing sequence (that is,* $S_0 \leq S_1 \leq S_2 \leq \ldots$*) or a nonincreasing sequence (that is,* $S_0 \geq S_1 \geq S_2 \geq \ldots$*).*[1] *Assume also that* $\{S_n\}$ *is a bounded sequence (that is, all the values live on some finite interval). Then* $\lim_{n\to\infty} S_n$ *exists.* ∎

Theorem (L4). *If* $\lim_{n\to\infty} S_n = S$ *and* f *is any function that is continuous at* S, *then* $\lim_{n\to\infty} f(S_n) = f(S)$. ∎

If in Theorem (L4) we take $f(x) = |x|$, we get the frequently-used fact that

$$\lim_{n\to\infty} S_n = S \Longrightarrow \lim_{n\to\infty} |S_n| = |S|.$$

The converse to this implication is false in general; but there is one case in which it is true:

Theorem (L5). *If* $\lim_{n\to\infty} |S_n| = 0$, *then* $\lim_{n\to\infty} S_n = 0$. ∎

Theorem (L6) (Sandwich Theorem). *If* $S_n \leq T_n \leq W_n$ *for all* n, *and if* $\lim_{n\to\infty} S_n = \lim_{n\to\infty} W_n = L$, *then* $\lim_{n\to\infty} T_n = L$ *as well.* ∎

Example 1. Let $-1 < x < 1$. Then $\lim_{n\to\infty} x^n = 0$.

Proof. First note that by (L5), I will be done if I can show that $\lim_{n\to\infty} |x^n| = 0$; and that is what I will do.[2]

Since $1 \geq |x| \geq |x|^2 \geq \cdots \geq 0$, the sequence $\{|x|^n\}$ is a nonincreasing bounded sequence. By (L3), then, it has some limit L. Moreover, replacing n with $n+1$ just starts the sequence at the first term instead of the 0^{th} term; this means the limit L will be unchanged if I do this. Then (by the theorems of this section) the following computation is valid:

$$L = \lim_{n\to\infty} |x|^{n+1} = \lim_{n\to\infty} |x| \cdot |x|^n = |x| \cdot \lim_{n\to\infty} |x|^n = |x| \cdot L.$$

But since $1 \neq |x|$, $L = |x| \cdot L$ implies $L = 0$. ∎

IV. First Theorems concerning Series

With the theorems of §**III**, I can now begin to tackle the theory of infinite series. First, let me discuss a simple but important example, which depends on an algebraic exercise, which I did for you in class.

Exercise 1. Show that if $x \neq 1$, then

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Hint: Multiply out $(1 - x) \cdot (1 + x + x^2 + \cdots + x^n)$.

Example 2. If $-1 < x < 1$, then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.$$

[1] Your text calls these *monotonic* sequences.
[2] Note that $|x^n| = |x|^n$.

Proof. By the exercise,

$$S_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x};$$

so

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \left(\sum_{k=0}^{n} x^k \right) = \lim_{n\to\infty} \left(\frac{1 - x^{n+1}}{1 - x} \right).$$

Since (by Example 1) $\lim_{n\to\infty} x^{n+1} = 0$, $\lim_{n\to\infty} (\frac{1 - x^{n+1}}{1 - x}) = \frac{1 - 0}{1 - x} = \frac{1}{1 - x}$. ∎

Theorem (S1). *If* $\sum_{k=0}^{\infty} a_k = S$ *and* $\sum_{k=0}^{\infty} b_k = T$, *then* $\sum_{k=0}^{\infty} c \cdot a_k = c \cdot S$ *and* $\sum_{k=0}^{\infty} (a_k \pm b_k) = S \pm T$. ∎

Example 2 and Theorem (S1) above actually tell you the values of certain infinite sums. Most of the theorems in this theory do not do this, as you will see starting in §**III**, below. For example, there does not exist a formula for $\sum_{k=0}^{\infty} (a_k \cdot b_k)$ in terms of $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, even if all three of these series converge.

The last theorem I will include in this section is used for showing that a sum does **NOT** converge. It says that in order for a series to converge, the terms you add must approach zero.

Theorem (S2) (n^{th} Term Test). *If* $\sum_{k=0}^{\infty} a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$. (*Equivalently: if it is not true that* $\lim_{k\to\infty} a_k = 0$, *then* $\sum_{k=0}^{\infty} a_k$ *does not converge.*) ∎

The converse to this theorem is false: there are many sequences $\{a_k\}$ such that $\lim_{k\to\infty} a_k = 0$ but $\sum_{k=0}^{\infty} a_k$ diverges anyway. In such cases, intuitively speaking, the terms of $\{a_k\}$ don't go to zero "fast enough." (Here is an example: $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge, even though $\lim_{k\to\infty} \frac{1}{k} = 0$. I will prove this in class twice, once by using a trick and once by applying a technique. The technique is the "integral test"—see the next section of this handout.)

When you are faced with the task of deciding whether a certain series converges, you should first apply the n^{th} Term Test to make sure the terms go to zero. If they don't, throw the series into the trash can; if they do, move on to other tests.

V. Series of Positive Terms

The theorems in this section deal with $\sum_{k=0}^{\infty} a_k$ in the case $a_k \geq 0$. These infinite sums can be compared in the same way that you have learned to compare improper integrals of positive functions. (The condition "$a_k \geq 0$" is less of a restriction than you might guess; see §**VI** below.)

Theorem (S3) (Comparison Test). *Suppose* $0 \leq a_k \leq b_k$. *If* $\sum_{k=0}^{\infty} b_k$ *converges, then* $\sum_{k=0}^{\infty} a_k$ *converges.* (*Equivalently: if* $\sum_{k=0}^{\infty} a_k$ *diverges, then* $\sum_{k=0}^{\infty} b_k$ *diverges.*)

Proof. Assume $\sum b_k$ converges; say $\sum_{k=0}^{\infty} b_k = T < \infty$. Denote the partial sums by

$$S_n = \sum_{k=0}^{n} a_k \qquad \text{and} \qquad T_n = \sum_{k=0}^{n} b_k.$$

Then, first of all

$$0 \le S_n \le T_n \;=\; \sum_{k=0}^{n} b_k \le \sum_{k=0}^{\infty} b_k = T < \infty,$$

so $\{S_n\}$ is a bounded sequence; and secondly, $a_k \ge 0 \Longrightarrow S_0 \le S_1 \le \cdots$, so $\{S_n\}$ is a nondecreasing sequence. By Theorem (L3), then, $\{S_n\}$ has a limit. This says that $\sum_{k=1}^{\infty} a_k$ converges. ∎

The Comparison Test says that you can show a series of nonnegative terms diverges by finding a smaller (nonnegative) divergent series, or you can show such a series converges by finding a larger convergent series. The following theorem, which follows directly from this one, also relates the convergence/divergence of one series to that of a second series.

Theorem (S4) (Limit Comparison Test). *Let $a_k \ge 0$ and $b_k > 0$, and suppose that*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0.$$

Then $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges. ∎

To use either of the theorems above, you must already know how one of the two series behaves in order to learn how the other behaves. By themselves, they are useless; but they become useful when applied in connection with theorems like the next one, the Integral Test. This theorem says that (under certain conditions) a series converges $\Longleftrightarrow$ a related improper integral converges. This fact is useful because the integrals are often easier to evaluate than partial sums are. An important example is given as a corollary to the theorem.

Theorem (S5) (Integral Test). *Suppose $f(x)$ is positive and decreasing for $x \ge n$. Then*

$$\sum_{k=n}^{\infty} f(k) \text{ and } \int_{n}^{\infty} f(x)dx$$

either both converge or both diverge. ∎

Corollary. *Let $p > 0$. The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges}, & \text{if } p > 1, \text{ and} \\ \text{diverges}, & \text{if } p \le 1. \end{cases} \quad ∎$$

VI. Absolute Convergence

Much of the usefulness of the theorems in the previous section comes from this theorem:

Theorem (S6) (Absolute Convergence Test). *If $\sum_{k=0}^{\infty} |a_k|$ converges then $\sum_{k=0}^{\infty} a_k$ converges.* ∎

This theorem gives a strategy that often works for showing convergence: Replace each term in a series with its absolute value, and then try to show that the infinite series of absolute values converges. I must put a warning in here, though: If the series of absolute values diverges, the original series might still converge. If the series of absolute values converges, the series is said to *converge absolutely*; if $\sum_{k=0}^{\infty} a_k$ converges but $\sum_{k=0}^{\infty} |a_k|$ diverges, then the series is said to *converge conditionally*. The Alternating Series Test—§**VIII**, below—can generate many examples of conditionally converging series. I'll give an example there.

VII. The Root Test and the Ratio Test

These two tests are very similar, and they are both used extensively in the theory of "infinite polynomials." Notice that both of them test absolute convergence, since they deal only with $|a_k|$.)

Theorem (S7) (Ratio Test). *Suppose that the following limit exists:*

$$M = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}.$$

Then

$$\begin{cases} M < 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \textit{ converges;} \\ M > 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \textit{ diverges;} \\ \text{if } M = 1, & \sum_{k=0}^{\infty} a_k \textit{ might either converge or diverge.} \end{cases} \qquad \blacksquare$$

Theorem (S8) (Root Test). *Suppose that the following limit exists:*

$$M = \lim_{k\to\infty} \sqrt[k]{|a_k|}.$$

Then

$$\begin{cases} M < 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \textit{ converges;} \\ M > 1 \Longrightarrow & \sum_{k=0}^{\infty} a_k \textit{ diverges;} \\ \text{if } M = 1, & \sum_{k=0}^{\infty} a_k \textit{ might either converge or diverge.} \end{cases} \qquad \blacksquare$$

VIII. The Alternating Series Test

This is the only test on this handout that can handle conditionally convergent series.

Theorem (S8) (Alternating Series Test). *If* $a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$, *and if* $\lim_{k\to\infty} a_k = 0$, *then the "alternating series"*

$$a_0 - a_1 + a_2 - a_3 + \cdots = \sum_{k=0}^{\infty} (-1)^k a_k$$

will converge. Moreover,

$$|S - S_n| \leq a_{n+1}. \quad \blacksquare$$

Example 4. The "alternating harmonic series"

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots$$

converges by this theorem; and since $\sum(1/k)$ diverges—see the corollary to Theorem (S5)—this convergence is conditional not absolute.