

## Summary of Tests for Convergence or Divergence of Series

### I. The fundamental definition.

For a sequence  $\{a_k\}$ , let  $s_n := \sum_{k=1}^n a_k$ . Then  $\sum_{k=1}^{\infty} a_k$  exists (and equals  $s$ ) precisely if

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n a_k \right) = s.$$

In other words,

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n a_k \right).$$

If this limit exists, one says that the series  $\sum_{k=1}^{\infty} a_k$  **converges**; if the limit does not exist, one says that the series  $\sum_{k=1}^{\infty} a_k$  **diverges**. The sequence  $\{s_n\}$  is called the **sequence of partial sums** of the series  $\sum_{k=1}^{\infty} a_k$ .

### II. Test for Divergence.

Unless  $\lim_{k \rightarrow \infty} a_k = 0$ ,  $\sum_{k=1}^{\infty} a_k$  diverges.

### III. Geometric series.

If  $-1 < r < 1$ , then  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ ; otherwise,  $\sum_{k=0}^{\infty} ar^k$  diverges.

### IV. Arithmetic with Convergent Series.

If  $\sum_{k=1}^{\infty} a_k = A$  and  $\sum_{k=1}^{\infty} b_k = B$ , then  $\sum_{k=1}^{\infty} (a_k \pm b_k) = A \pm B$ , and  $\sum_{k=1}^{\infty} ca_k = cA$ , for any number  $c$ .

### V. The Integral Test and $p$ -series.

Suppose  $f(x)$  is positive and decreasing for  $x \geq a$ . Then

$$\sum_{k=a}^{\infty} f(k) \text{ and } \int_a^{\infty} f(x) dx$$

either both converge or both diverge. Moreover, if they both converge, then for any  $n \geq a$ ,

$$\int_{n+1}^{\infty} f(x) dx \leq s - s_n \leq \int_n^{\infty} f(x) dx.$$

**An application:** Let  $p > 0$ . The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges,} & \text{if } p > 1, \text{ and} \\ \text{diverges,} & \text{if } p \leq 1. \end{cases}$$

### VI. Comparison tests

**Plain Vanilla Comparison Test.** Suppose  $0 \leq a_k \leq b_k$ . If  $\sum_{k=0}^{\infty} b_k$  converges, then  $\sum_{k=0}^{\infty} a_k$  converges.

Equivalently: If  $\sum_{k=0}^{\infty} a_k$  diverges, then  $\sum_{k=0}^{\infty} b_k$  diverges. Moreover: if both series converge, then for any  $n$ ,

$$|s - s_n| \leq |t - t_n|,$$

where  $\{s, s_n\} \longleftrightarrow \sum a_k$  and  $\{t, t_n\} \longleftrightarrow \sum b_k$ .

**Limit Comparison Test.** Let  $a_k \geq 0$  and  $b_k > 0$ , and suppose that

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = c > 0.$$

Then  $\sum_{k=0}^{\infty} a_k$  converges if and only if  $\sum_{k=0}^{\infty} b_k$  converges.<sup>1</sup>

### VII. Absolute convergence.

If  $\sum_{k=0}^{\infty} |a_k|$  converges then  $\sum_{k=0}^{\infty} a_k$  converges.

In this case, one says that the series  $\sum_{k=0}^{\infty} a_k$  **converges absolutely**; if  $\sum_{k=0}^{\infty} a_k$  converges while  $\sum_{k=0}^{\infty} |a_k|$  diverges,

then one says that  $\sum_{k=0}^{\infty} a_k$  **converges conditionally**. In the case of absolutely convergent series, any rearrangement of the terms produces a new series that converges to the same sum as did the old one. For a conditionally convergent series, rearrangements can produce new series whose sequences partial sums  $\{s_n\}$  will behave in almost any prescribed fashion whatsoever.

### VIII. Alternating series.

If  $b_0 \geq b_1 \geq b_2 \geq \dots \geq 0$ , and if  $\lim_{k \rightarrow \infty} b_k = 0$ , then the “alternating series”

$$b_0 - b_1 + b_2 - b_3 + \dots = \sum_{k=0}^{\infty} (-1)^k b_k$$

will converge. Moreover,

$$|s - s_n| \leq b_{n+1}.$$

It is worth noting that of all the tests on this sheet, **ONLY** this one—the Alternating Series Test—can detect conditional convergence. It is also important that you apply this test to  $\sum a_k$  **ONLY AFTER** after having considered  $\sum |a_k|$ .

### IX. The Root and Ratio Tests.

**Ratio Test.** Suppose that the following limit exists:

$$L = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}.$$

Then

$$\left\{ \begin{array}{l} L < 1 \implies \sum_{k=0}^{\infty} a_k \text{ converges absolutely;} \\ L > 1 \implies \sum_{k=0}^{\infty} a_k \text{ diverges;} \\ \text{if } L = 1, \quad \sum_{k=0}^{\infty} a_k \text{ might converge absolutely, converge conditionally, or diverge.} \end{array} \right.$$

**Root Test.** Suppose that the following limit exists:

$$L = \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}.$$

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<sup>1</sup> If limit  $c = 0$  and  $\sum b_k$  converges, then  $\sum a_k$  converges. However, if  $c = 0$  and  $\sum b_k$  diverges, then  $\sum a_k$  might either converge or diverge.

Then

$$\left\{ \begin{array}{l} L < 1 \implies \sum_{k=0}^{\infty} a_k \text{ converges absolutely;} \\ L > 1 \implies \sum_{k=0}^{\infty} a_k \text{ diverges;} \\ \text{if } L = 1, \quad \sum_{k=0}^{\infty} a_k \text{ might converge absolutely, converge conditionally, or diverge.} \end{array} \right.$$

**X. A final word.**

It is essential that you pay careful attention to the precise statements of these tests: what conditions you need to apply them and what you may and may not conclude after applying them. In particular, four of them—the Test for Divergence, the Limit Comparison Test, and the Root and Ratio Tests—involve taking limits. Do not forget, while taking one of these limits, what test you are applying and therefore what conclusions you will be allowed to draw.