Derivations of Some of the More Important Trig Identities

This handout derives many of the most frequently occurring Trig identities. As you will see, we will have to work hard to derive one of them, but then the others, if introduced in the correct order, will follow as easy consequences. I will use without proving them here identities (1)–(5), although I will discuss them briefly.

$$\sin^2(x) + \cos^2(x) = 1 \tag{1}$$

$$\tan^2(x) + 1 = \sec^2(x) \tag{2}$$

$$\cot^2(x) + 1 = \csc^2(x) \tag{3}$$

$$\cos(-x) = \cos(x) \tag{4}$$

$$\sin(-x) = -\sin(x) \tag{5}$$

This is the identity whose proof will make us work:

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \tag{6}$$

Proof. Consider the diagram below, in which angles of sizes x, y, and $x - y$ have been drawn on a unit circle.

||
|| $A = \big(\cos(x), \sin(x)\big)$
||
|| $B = \big(\cos(y), \sin(y)\big)$
||
|| $C = \big(\cos(x - y), \sin(x - y)\big)$
||
|| $D = (1, 0)$
||
||
||

(The drawing assumes $x \geq y \geq 0$, but—by equation (4)—once we know the identity for $x \geq y \geq 0$, we will have it for all values of x and y.) Observe that we can write the coordinates of the points A, B, C, and D as shown in the diagram; and that (since the two shaded sectors are congruent), the distance d_1 from A to B will equal the distance d_2 from C to D. The identity we want will flow from this fact:

$$d_1^2 = d_2^2$$

$$\big(\cos(x) - \cos(y)\big)^2 + \big(\sin(x) - \sin(y)\big)^2 = \big(\cos(x - y) - 1\big)^2 + \sin^2(x - y)$$

(square out:)

$$\big[\cos^2(x) - 2\cos(x)\cos(y) + \cos^2(y)\big] + \big[\sin^2(x) - 2\sin(x)\sin(y) + \sin^2(y)\big] =$$

$$\big[\cos^2(x - y) - 2\cos(x - y) + 1\big] + \sin^2(x - y).$$

Now using identity (1) three times gives

$$1 + 1 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y) = 1 + 1 - 2\cos(x - y).$$

Now subtract 2 from both sides; and then multiply both sides by $-\frac{1}{2}$. Identity (6) has appeared. ∎

Consequences. The remaining identities follow easily now.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{7}$$

Proof. **Exercise 1.**

$$\cos(\frac{\pi}{2} - x) = \sin(x) \tag{8}$$

Proof. **Exercise 2.** (Use equation (6).)

$$\sin(\frac{\pi}{2} - x) = \cos(x) \tag{9}$$

Proof. Since $x = \big(\frac{\pi}{2} - (\frac{\pi}{2} - x)\big)$, we have

$$\cos(x) = \cos\big(\frac{\pi}{2} - (\frac{\pi}{2} - x)\big) \underset{\substack{\uparrow \\ (8)}}{=} \sin(\frac{\pi}{2} - x). \blacksquare$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \tag{10}$$

Proof.

$$\sin(x+y) \underset{\substack{\uparrow \\ (8)}}{=} \cos\big(\frac{\pi}{2} - (x+y)\big) = \cos\big((\frac{\pi}{2} - x) - y\big);$$

by equation (6), this equals

$$\cos(\frac{\pi}{2} - x)\cos(y) + \sin(\frac{\pi}{2} - x)\sin(y) \underset{\substack{\uparrow \\ (8,9)}}{=} \sin(x)\cos(y) + \cos(x)\sin(y). \blacksquare$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) \tag{11}$$

Proof. **Exercise 3.**

It is worth noting that identities (6), (7), (10) and (11) are often summarized:

$$\begin{cases} \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y); \\ \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y). \end{cases}$$

$$\sin(2x) = 2\sin(x)\cos(x); \quad \cos(2x) = \cos^2(x) - \sin^2(x) \tag{12, 13}$$

$$\sin(x+\pi) = -\sin(x); \quad \cos(x+\pi) = -\cos(x); \quad \tan(x+\pi) = \tan(x) \tag{14–16}$$

Proofs. **Exercises 4–8.**

$$\sin(x + \frac{\pi}{2}) = \cos(x) \tag{17}$$

Proof. **Exercise 9.**

Summary Sheet

Here is a list of the identities, reprinted from *pp.* 1–3. In my view, are all worth memorizing. There are many others that come up slightly less frequently, but these others can all be easily derived from the ones discussed here.

$$\sin^2(x) + \cos^2(x) = 1 \tag{1}$$

$$\tan^2(x) + 1 = \sec^2(x) \tag{2}$$

$$\cot^2(x) + 1 = \csc^2(x) \tag{3}$$

$$\cos(-x) = \cos(x) \tag{4}$$

$$\sin(-x) = -\sin(x) \tag{5}$$

$$\begin{cases} \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y); \\ \\ \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y). \end{cases} \tag{6, 7, 10, 11}$$

$$\sin(2x) = 2\sin(x)\cos(x); \quad \cos(2x) = \cos^2(x) - \sin^2(x) \tag{12, 13}$$

$$\sin(x + \pi) = -\sin(x); \quad \cos(x + \pi) = -\cos(x); \quad \tan(x + \pi) = \tan(x) \tag{14–16}$$

$$\cos(\frac{\pi}{2} - x) = \sin(x) \tag{8}$$

$$\sin(\frac{\pi}{2} - x) = \cos(x) \tag{9}$$

$$\sin(x + \frac{\pi}{2}) = \cos(x) \tag{17}$$