Worksheet: first and last step of trig sub computations

First step practice. Convert each of the integrals below to the form $\int^{f(x)} g(\theta)\, d\theta$.

$$\int (6-x^2)^{3/2}\, dx$$

$$\int x^2\sqrt{x^2-9}\, dx$$

$$\int \frac{\sqrt{7+5x^2}}{x}\, dx$$

$$\int (x^2+4x)^{3/2}\, dx$$

(Hint: complete the square.)

Last step practice. Eliminate all trig and inverse trig functions.

$$\frac{\sec^3(\theta)\tan(\theta)}{4} + \frac{3\sec(\theta)\tan(\theta)}{8} + \frac{3}{8}\ln\left|\sec(\theta)+\tan(\theta)\right|\Bigg|^{\theta=\tan^{-1}(x/3)}$$

$$\cos(\theta)+\sin^2(\theta)\Bigg|^{\theta=\sin^{-1}\left(\frac{2x}{5}\right)}$$

Answers

$$\int (6-x^2)^{3/2}\,dx$$

$$x = \sqrt{6}\sin(\theta)$$
$$dx = \sqrt{6}\cos(\theta)d\theta$$
$$\theta = \sin^{-1}\left(\tfrac{x}{\sqrt{6}}\right)$$

$$\int^{\sin^{-1}\left(\frac{x}{\sqrt{6}}\right)} \left(6-6\sin^2(\theta)\right)^{3/2}\sqrt{6}\cos(\theta)\,d\theta$$

$$\int x^2\sqrt{x^2-9}\,dx$$

$$x = 3\sec(\theta)$$
$$dx = 3\sec(\theta)\tan(\theta)d\theta$$
$$\theta = \sec^{-1}\left(\frac{x}{3}\right)$$

$$\int^{\sec^{-1}\left(\frac{x}{3}\right)} 9\sec^2(\theta)\left(\sqrt{9\sec^2(\theta)-9}\right)3\sec(\theta)\tan(\theta)\,d\theta$$

$$\int \frac{\sqrt{7+5x^2}}{x}\,dx$$

$$5x^2 = 7\tan^2(\theta)$$
$$x = \sqrt{\frac{7}{5}}\tan(\theta)$$
$$dx = \sqrt{\frac{7}{5}}\sec^2(\theta)d\theta$$
$$\theta = \tan^{-1}\left(\sqrt{\frac{5}{7}}x\right)$$

$$\int^{\tan^{-1}\left(\sqrt{\frac{5}{7}}x\right)} \frac{\sqrt{7+7\tan^2(\theta)}}{\sqrt{\frac{7}{5}}\tan(\theta)}\sqrt{\frac{7}{5}}\sec^2(\theta)\,d\theta$$

$$\int (x^2+4x)^{3/2}\,dx = \int \left((x+2)^2-4\right)^{3/2}\,dx$$

$$x+2 = 2\sec(\theta)$$
$$dx = 2\sec(\theta)\tan(\theta)d\theta$$
$$\theta = \sec^{-1}\left(\frac{x+2}{2}\right)$$

$$\int^{\sec^{-1}\left(\frac{x+2}{2}\right)} \left(4\sec^2(\theta)-4\right)^{3/2}2\sec(\theta)\tan(\theta)\,d\theta$$

Last step practice. Eliminate all trig and inverse trig functions.

$$\left.\frac{\sec^3(\theta)\tan(\theta)}{4} + \frac{3\sec(\theta)\tan(\theta)}{8} + \frac{3}{8}\ln\left|\sec(\theta)+\tan(\theta)\right|\right|^{\theta=\tan^{-1}(x/3)}$$

$$= \frac{1}{4}\left(\frac{\sqrt{9+x^2}}{3}\right)^3\frac{x}{3} \;+\; \frac{3}{8}\left(\frac{\sqrt{9+x^2}}{3}\right)\frac{x}{3} \;+\; \frac{3}{8}\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|,$$

because $\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9+x^2}}{3}$.

$$\left.\cos(\theta)+\sin^2(\theta)\right|^{\theta=\sin^{-1}\left(\frac{2x}{5}\right)}$$

$$= \frac{\sqrt{25-4x^2}}{5} \;+\; \left(\frac{2x}{5}\right)^2,$$

because $\cos\left(\sin^{-1}\left(\frac{2x}{5}\right)\right) = \frac{\sqrt{25-4x^2}}{5}$.