Notes from a talk given by Peter Winkler
(greatly expanded)

1) Context: (A) Family of structures; each structure has an "order" n $(0,1,2,\dots)$. (At most) finitely many structures of each order. Let $f(n) = \#$ structures of order n.

Ex: Permutations of $\{1,2,\dots,n\}$; partitions of $\{1,\dots,n\} = [n]$; graphs with vertex set $\{1,\dots,n\}$. In each case, the order is n.

For permutations: $f(n) = n!$

For partitions: $f(n)$ ("b_n") is $\frac{1}{e}\sum_{r=0}^{\infty}\frac{r^n}{r!}$ (!!)

For graphs: $f(n) = 2^{\binom{n}{2}}$.

(B) A property enjoyed by some of the structures (i.e. a subset of the structures). Examples:

For permutations of order n: $(\forall i \in [n])(\sigma(i) \neq i)$

(σ is fixed-point-free)

For partitions of order n: $(\forall i \in [n])(\exists j \in [n])(i \neq j \text{ and } i \equiv_{\pi} j)$

(π is singleton-free)

For graphs of order n: $\rightarrow (\forall i \in [n])(\exists j \in [n])(i \neq j \text{ and } \mathrm{Edge}_G(i,j))$

(G is isolated-point-free)

$\rightarrow$ G is connected.

Observe that all of the properties except the last are expressed by a closed $\mathcal{L}$-wf whose variables range over $\mathbb{N}$. These are called *first-order properties*. (Winkler states that connectedness is provably not first-order.) We will limit ourselves to first-order properties.

Given a family of structures, let ϕ be a closed $\mathcal{L}$-wf that is being interpreted as the definition of a property. Define

$$\operatorname{Limprob}(\phi) := \lim_{n \to \infty} \frac{\#\text{objects of order } n \text{ having property } \phi}{f(n)}$$

(if this limit exists)

Example ("Hatcheck Problem")

$$\lim_{n \to \infty} \frac{\#\text{fixed-point-free permutations of } [n]}{n!} = \frac{1}{e}.$$

I don't know formula for

$$\lim_{n \to \infty} \frac{\#\text{ singleton-free partitions of } [n]}{b_n},$$ but this

may be do-able: If f_n is the numerator and

$$F(x) = \sum f_n \frac{x^n}{n!}, \quad \text{I get } F(x) = e^{e^x - 1 - x}$$

(Dmytro, please check!)

II Random Graphs: To make a random graph of order n, flip a fair coin $\binom{n}{2}$ times to decide whether to include or exclude each possible edge. For a given first-order property ϕ, then, the probability that a random graph has property ϕ is just

$$\frac{\#\text{graphs of order } n \text{ with property } \phi}{2^{\binom{n}{2}}},$$

So $\text{Limprob}(\phi)$ can be interpreted as the "probability that a random graph has property ϕ".

Mathematicians Erdős and Renyi observed that frequently in these cases, either $\text{Limprob}(\phi) = 1$ or $\text{Limprob}(\phi) = 0$. Further, they knew of no cases to the contrary, so they conjectured that this would always be true. The conjecture was proved in 1976.

Theorem (Ronald Fagan, 1976). For any first-order property ϕ of graphs,

$$\text{either } \text{Limprob}(\phi) = 1$$

$$\text{or } \quad \text{Limprob}(\phi) = 0$$

"Zero-one law".

The proof will be divided into several steps

<u>Proof, step 1</u>: The "Alice's Restaurant" predicates.

For each $i \geq 0$, $j \geq 0$, let ϕ_{ij} be the closed wf

$$(\forall x_1)\cdots(\forall x_i)(\forall y_1)\cdots(\forall y_j)(\exists z)\left(edge(z,x_1)\,\&\cdots\&\,edge(z,x_i)\,\&\sim edge(z,y_1)\,\&\cdots\&\sim edge(z,y_j)\right)$$

(We are working in an $\mathcal{L}$ set up for graph theory. $\underbrace{\text{It has pred. letter}}_{\text{(call it GC)}}$ "edge". There are some axioms, such as (I suppose) $(\forall x)(\forall y)(edge(x,y) \rightarrow x \neq y)$. Any graph is an interpretation I_G; closed wf ϕ "holds" for graph G if $I_G \models \phi$.)

<u>Lemma 1</u>. For each i, j, $\mathrm{Limprob}(\phi_{ij}) = 1$.

Pf. First, choose $\overbrace{}^{i+j+1}$ integers $\{x_1,\ldots,x_i\}$, $\{y_1,\ldots,y_j\}$, and $\{z\}$. Observe that the number of graphs of order n in which all x's are adjacent to z and no y's are adjacent to z is $2^{\binom{n}{2}-i-j}$,

So the probability that a random graph has this property is:

$$\frac{2^{\binom{n}{2}-i-j}}{2^{\binom{n}{2}}} = \left(\frac{1}{2}\right)^{i+j}.$$

Next: For this same choice of $x_1,\ldots,x_i$, $y_1,\ldots,y_j$, the probability of choosing a random graph for which every z fails is:

$$\underbrace{\left(1-\left(\tfrac{1}{2}\right)^{i+j}\right)}_{\text{call this number } \alpha}{}^{n-i-j}$$ (by independence, can multiply across the $n-i-j$ possible z's).

$$(\alpha)^{n-i-j}$$

Third, for each graph G of order n and each choice of disjoint subsets $S = \{x_1, \dots, x_i\}$ and $T = \{y_1, \dots, y_j\}$, let

$$Y_n^{S,T}(G) = \begin{cases} 1 & \text{if every } z \text{ fails for } S, T \text{ in } G \\ 0 & \text{else.} \end{cases}$$

This is a random variable on the (uniform) sample space of graphs of order n

A routine computation shows that

$$E\left(Y_n^{S,T}(G)\right) = \alpha^{n-i-j}$$

Fourth, for each graph G of order n, let $X_n^{i,j}(G)$ be the number of sets choices S, T for which every z fails.

Since

$$X_n^{ij} = \sum_{S,T} Y_n^{S,T},$$ by additivity of expected value,

we get

$$E\left(X_n^{ij}\right) = \binom{n}{i}\binom{n-i}{j}\alpha^{n-i-j}$$

$$= \frac{1}{\alpha^{i+j}}\binom{n}{i}\binom{n-i}{j}\alpha^{n}.$$

Now, $$\binom{n}{i}\binom{n-i}{j} = \frac{n(n-1)\cdots(n-i-j+1)}{i!\,j!}$$ is a poly in n of degree $(i+j)$,

and $\alpha = |\alpha| < 1$, so $$\lim_{n\to\infty}\binom{n}{i}\binom{n-i}{j}\alpha^n = 0.$$

Thus, $$\lim_{n\to\infty} E\left(X_n^{ij}\right) = 0. \quad (*)$$

Fifth, observe that

$$E(X_n^{ij}) = \sum_{k\geq 0} k \cdot P(X_n^{ij}=k) \geq \sum_{k\geq 1} P(X_n^{ij}=k) = P(X_n^{ij}\geq 1).$$

So by ⊛,

$$\lim_{n\to\infty} P(X_n^{ij}\geq 1) = 0, \text{ so } \lim_{n\to\infty} P(X_n^{ij}=0) = 1.$$

But this last is $\text{Limprob}(\phi_{ij})$ QED (Lemma 1).

<u>Proof, step 2.</u> Let $\phi_1, \ldots, \phi_k$ be some k of the $\{\phi_{ij}\}$; let GC be the first-order system set up for graph theory.

<u>Lemma 2</u> $GC \cup \{\phi_1, \ldots, \phi_k\}$ is consistent.

Pf. Let P_n be the probability that a randomly-chosen graph of order n will model this system. Then

$$1 \geq P_n = P(I_G \models \phi_1 \text{ and } \cdots \text{ and } I_G \models \phi_k) = 1 - P(I_G \not\models \phi_1 \text{ OR} \cdots \text{OR } I_G \not\models \phi_k)$$

$$\left(P(A_1 \cup \cdots \cup A_k) \leq P(A_1) + \cdots + P(A_k) \Rightarrow\right) \geq 1 - \sum_{i=1}^{k} P(I_G \not\models \phi_i);$$

Take $\lim\limits_{n\to\infty}$: $\lim\limits_{n\to\infty}\left(1 - \sum_{i=1}^{k} P(I_G \not\models \phi_i)\right) = 1 - \sum_{i=1}^{k} \lim\limits_{n\to\infty} P(I_G \not\models \phi_i)$

$$= 1 - \sum_{i=1}^{k} \text{Limprob}(\sim\phi_i) = 1.$$

By Squeeze Theorem, then, $\boxed{\lim_{n\to\infty} P_n = 1}$ So for some n, $P_n > 0$; one can find a graph of order n that models $GC \cup \{\phi_1, \ldots, \phi_k\}$

QED (Lemma 2).

Proof, Step 3. Let $J = GC \cup \{\phi_{ij} : i \geq 0, j \geq 0\}$. J is a consistent extension of GC.

Proof: Compactness Theorem.

Proof, Step 4. Let G and H be two denumerable models of J. Then, as graphs, G and H are isomorphic.

Proof Enumerate the vertices of G : $g_0, g_1, g_2, \ldots$
H : $h_0, h_1, h_2, \ldots$

I will construct the isomorphism $f : G \to H$ by iteratively re-enumerating the vertices $\left\{ \begin{matrix} \hat{g}_0, \hat{g}_1, \ldots \\ \hat{h}_0, \hat{h}_1, \ldots \end{matrix} \right\}$ and making $f(\hat{g}_i) = \hat{h}_i$.

Initially: put $\hat{g}_0 = g_0$, $\hat{h}_0 = h_0$, and $f(\hat{g}_0) = \hat{h}_0$: $\begin{matrix} \hat{g}_0 \\ \updownarrow \\ \hat{h}_0 \end{matrix}$.

Then, put $\hat{g}_1 = g_1$, and let $\hat{h}_1$ be the first vertex of H that is/is not adjacent to $\hat{h}_0$ according as $\hat{g}_1$ is/is not adjacent to $\hat{g}_0$.
We have: $\begin{matrix} \hat{g}_0 & \sqsubset\!\sqsupset & \hat{g}_1 \\ | & & | \\ \hat{h}_0 & \sqsubset\!\sqsupset & \hat{h}_1 \end{matrix}$ (where $\sqsubset\!\sqsupset$ match).

Now, let $\hat{h}_2$ be the first unprocessed h : $\hat{h}_0 \sqsubset\!\sqsupset \hat{h}_1 \sqsubset\!\sqsupset \hat{h}_2$;

Let $\hat{g}_2$ be the first unprocessed g so that $\hat{g}_0 \sqsubset\!\sqsupset \hat{g}_1 \sqsubset\!\sqsupset \hat{g}_2$
boxes match.

continued

Proceed iteratively. After n steps, have

$$\begin{array}{ccc} \hat{g}_0 & \cdots & \hat{g}_n \\ \updownarrow & \cdots & \updownarrow \\ \hat{h}_0 & \cdots & \hat{h}_n \end{array}$$

and the finite graphs displayed are isomorphic.

When you finish, you will have processed all g's and h's. (This is guaranteed by alternately choosing from G and H.) If this map failed to be an isomorphism (say $\hat{g}_i$ is adjacent to $\hat{g}_j$ but $\hat{h}_i$ is not adjacent to $\hat{h}_j$), then this failure would have shown up in Stage $n = \max(i,j)$, contradiction.

Proof, Step 5. J is a complete first-order system.

Proof: If not $\vdash_J A$ and not $\vdash_J \neg A$ for some closed wf A, then we would have two first-order systems $J \cup \{A\}$ and $J \cup \{\neg A\}$ that are both consistent and so both have denumerable models. But as graphs, these models cannot be isomorphic, since wf A is true in one and false in the other.

Proof, final step. Let ϕ be any closed wf. Say $\vdash_J \phi$. The deduction is finite in length, so (see step 2), $\vdash_{GC \cup \{\phi_1, \ldots, \phi_k\}} \phi$, for some subset $\{\phi_1, \ldots, \phi_k\}$ of the $\{\phi_{i,j}\}$. As proved in Step 2,

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{\#\text{graphs of order } n \text{ that model } GC \cup \{\phi_1,\ldots,\phi_k\}}{2^{\binom{n}{2}}} = 1.$$

But for any graph G that models $GC \cup \{\phi_1, \ldots, \phi_k\}$, $I_G \models \phi$; so $\frac{\#\text{graphs of order } n \text{ that model } \phi}{2^{\binom{n}{2}}} \geq P_n$.

Hence $\lim prob\ \phi = \lim_{n\to\infty} \frac{\#\text{graphs order } n \text{ that model } \phi}{2^{\binom{n}{2}}} = 1$.

(If $\vdash_J \neg\phi$, then $\lim prob(\neg\phi) = 1$, so $\lim prob(\phi) = 0$. ∎