

More study guides for Test 3

Cheat Sheets. You can make yourself a one-page “cheat sheet” (standard letter paper, front and back). Here are some of the instructions from the test, so you can get an idea of which results you should know.

Directions. There are several results and theorems you will need to use repeatedly on this test. You may use these theorems freely without proof, but you must indicate where they are used. You can use these abbreviations:

TI	=	Triangle inequality
MIIM	=	(Modulus of the integral) \leq (integral of the modulus) (p. 114)
UBM	=	Result on the upper bound of the modulus of a contour integral (p. 130)
ANTI	=	Antiderivative Theorem (p. 135)
CGT	=	Cauchy-Goursat Theorem (p. 144) and its generalization (p. 149)
SCT	=	“Swiss Cheese Theorem” relating integrals on closed curves to integrals around interior closed curves, and its corollary, the principle of deformation of paths (p. 151)
CIF	=	Cauchy’s “Magic” Integral Formula (p. 157), including its generalization to the n th derivatives of f (p. 161)
TAY	=	Taylor’s Theorem (p. 182)
LAUR	=	Laurent’s Theorem (p. 190)
CONV	=	Result on the radius of convergence of Taylor series (p. 209)
RES	=	Residue Theorem (p. 225)

Other things. I recommend that you study homeworks 9, 10, 11, and 12, the sample problems I gave you last week, the treasure hunt problems, and any examples we did in class. You should be familiar with all the main theorems named above.

If you haven’t had enough practice, here are some more problems to try...

1. Use contour integration to show that the integral of $f(z) = \bar{z}$ around any circle in the complex plane (taken counterclockwise) is equal to $2i$ times the area of that circle.
2. Find the integral of $f(z) = \text{P.V.} z^{1/3} \sin(-e^{z^2})$ counterclockwise around the circle of radius 1 centered at $2 + 2i$.
3. Let f be a continuous (but not necessarily analytic) function such that $|f(z)| > 3$ on the unit circle. Explain how we know that $|\int_C 1/f(z) dz| < 2\pi/3$, where C is the unit circle, taken in the positive sense.

4. Prove that $f(z) = 1/z$ does not have an antiderivative on the domain $\mathbb{C} - \{0\}$ (hint: evaluate its integral around the unit circle, and use the antiderivative theorem).
5. Prove that $f(z) = 1/z$ does have an antiderivative on the upper half plane $\text{Im } z > 0$.
6. Find the Laurent series representations of

$$f(z) = \frac{1}{z(1-z)}, \quad z \neq 0, z \neq 1$$

- (a) centered at 0,
- (b) centered at 1, and
- (c) for the domain $|z| > 1$.

Indicate the largest annulus on which each series converges. In addition, use your Laurent series to find $\text{Res}_{z=0}f(z)$ and $\text{Res}_{z=1}f(z)$. What does this tell you about the integral of $f(z)$ around the circle $|z| = 2$, taken in the positive sense?

7. Find the Laurent series representation of

$$f(z) = \frac{\sin z}{z}, \quad z \neq 0$$

centered at 0, and indicate the largest annulus on which this series converges.¹ How can we define $f(0)$ so that $f(z)$ is entire (hint: look at the Laurent series)? In addition, use your Laurent series to find $\text{Res}_{z=0}f(z)$. What does this tell you about the integral of $f(z)$ around the circle $|z| = 2$, taken in the positive sense?

8. Explain the proof of lemma, p. 165.
9. Explain inequality (8), p. 148.
10. Explain inequality (10), p. 149.
11. Explain equation (8), p. 185.
12. Explain the next statement after (11), p. 184.
13. Explain the bottom statement, p. 184.

¹You can use the fact that $\sin z = z - z^3/3! + z^5/5! - \dots$.