More solutions to more problems for Test 3

1. Use contour integration to show that the integral of $f(z) = \overline{z}$ around any circle in the complex plane (taken counterclockwise) is equal to $2i$ times the area of that circle.

   Solution. The circle $|z - z_0| = R$ can be parametrized $z(t) = z_0 + Re^{it}$, for $0 \le t \le 2\pi$. Then $\int_C \overline{z}\, dz = \int_0^{2\pi} \overline{z_0 + Re^{it}} \cdot iRe^{it}\, dt = \int_0^{2\pi} \overline{z_0} iRe^{it} + iR^2\, dt = \overline{z_0}Re^{it} + tiR^2\big|_0^{2\pi} = 2\pi i R^2$

2. Find the integral of $f(z) = \text{P.V.}z^{1/3} \sin(-e^{z^2})$ counterclockwise around the circle of radius 1 centered at $2 + 2i$.

   Solution. Since the contour does not intersect the negative real line or the origin, P.V. $z^{1/3}$ is analytic inside and on the contour. All the other functions used to compose $f(z)$ are entire, so $f(z)$ is analytic inside and on the contour. Therefore, the contour integral is 0.

3. Let f be a continuous (but not necessarily analytic) function such that $|f(z)| > 3$ on the unit circle. Explain how we know that $\left|\int_C 1/f(z)\, dz\right| < 2\pi/3$, where C is the unit circle, taken in the positive sense.

   Solution. Since $f(z) \neq 0$ on the unit circle, $1/f(z)$ is defined and continuous on the circle, and $0 < |1/f(z)| < 1/3$. Using UBM, $\int_C |1/f(z)|\, dz < (1/3)L$, where L is the length of C, which is 2π. Using MIIM, $|\int_C 1/f(z)\, dz| \le \int_C |1/f(z)|\, dz < 2\pi/3$.

4. Prove that $f(z) = 1/z$ does not have an antiderivative on the domain $\mathbb{C} - \{0\}$ (hint: evaluate its integral around the unit circle, and use the antiderivative theorem).

   Solution. By the CIF, $\int_C 1/z\, dz = 2\pi i$, so at least one integral around a closed contour in the domain is nonzero. By ANTI, $1/z$ does not have an antiderivative throughout $\mathbb{C} - \{0\}$.

5. Prove that $f(z) = 1/z$ does have an antiderivative on the upper half plane $\text{Im}\, z > 0$.

   Solution. The upper half plane is simply connected, and $f(z)$ is analytic there (it's the quotient of nonzero analytic functions). By CGT, all the integrals around closed contours are zero, and hence by ANTI $f(z)$ has an antiderivative.

6. Find the Laurent series representations of

$$f(z) = \frac{1}{z(1-z)}, \; z \neq 0, z \neq 1$$

   (a) centered at 0,

   (b) centered at 1, and

   (c) for the domain $|z| > 1$.

   Indicate the largest annulus on which each series converges. In addition, use your Laurent series to find $\text{Res}_{z=0} f(z)$ and $\text{Res}_{z=1} f(z)$. What does this tell you about the integral of $f(z)$ around the circle $|z| = 2$, taken in the positive sense?

Solution.

(a) We want to expand in powers of z, so use

$$f(z) = (1/z) \cdot (-1 - z - z^2 - z^3 - \cdots) = -z^{-1} - 1 - z - z^2 - \cdots.$$

The radius of convergence is the distance to the nearest singular point, so it converges for $0 < |z| < 1$.

(b) We want to expand in powers of $z - 1$, so use

$$\frac{1}{z-1} \cdot \frac{1}{1+(z-1)} = \frac{1}{z-1}(1-(z-1)+(z-1)^2-\cdots) = (z-1)^{-1}-1+(z-1)-(z-1)^2+\cdots$$

The radius of convergence is the distance to the nearest singular point, so it converges for $0 < |z-1| < 1$.

(c) We want to expand in powers of $1/z$, so use

$$\frac{1}{z} \cdot \frac{-1/z}{1-1/z} = z^{-1}(-z^{-1} - z^{-2} - \cdots) = -z^{-2} - z^{-3} - z^{-4} - \cdots,$$

which converges when $0 < |1/z| < 1$.

Now, $\mathrm{Res}_{z=0} f(z) = -1$ is the coefficient of z^{-1} in the expansion around 0, and $\mathrm{Res}_{z=1} f(z) = 1$ is the coefficient of $(z-1)^{-1}$ in the expansion around 1. By RES, the integral of $f(z)$ around the circle of radius 2 is $2\pi i$ times the sum of the residues inside the circle, so the integral is 0.

7. Find the Laurent series representation of

$$f(z) = \frac{\sin z}{z}, \ z \neq 0$$

centered at 0, and indicate the largest annulus on which this series converges.[1] How can we define $f(0)$ so that $f(z)$ is entire (hint: look at the Laurent series)? In addition, use your Laurent series to find $\mathrm{Res}_{z=0} f(z)$. What does this tell you about the integral of $f(z)$ around the circle $|z| = 2$, taken in the positive sense?

Solution. We have $(\sin z)/z = 1 - z^2/3! + z^4/5! - \cdots$ which converges on the entire complex plane. This suggests that we can make $f(z)$ continuous by defining $f(0) \equiv 1$. The residue $\mathrm{Res}_{z=0} f(z)$ is 0 because there is no z^{-1} term in the Laurent series, so the integral around the circle is 0.[2]

For questions 8–13, write the abbreviation of the theorem that is used.

[1] You can use the fact that $\sin z = z - z^3/3! + z^5/5! - \cdots$.

[2] This discussion shows that we can extend $f(z)$ to an entire function by defining $f(0) \equiv 1$. The singularity at $z = 0$ is called a "removable" singularity.