Teaching the Mathematics of Music

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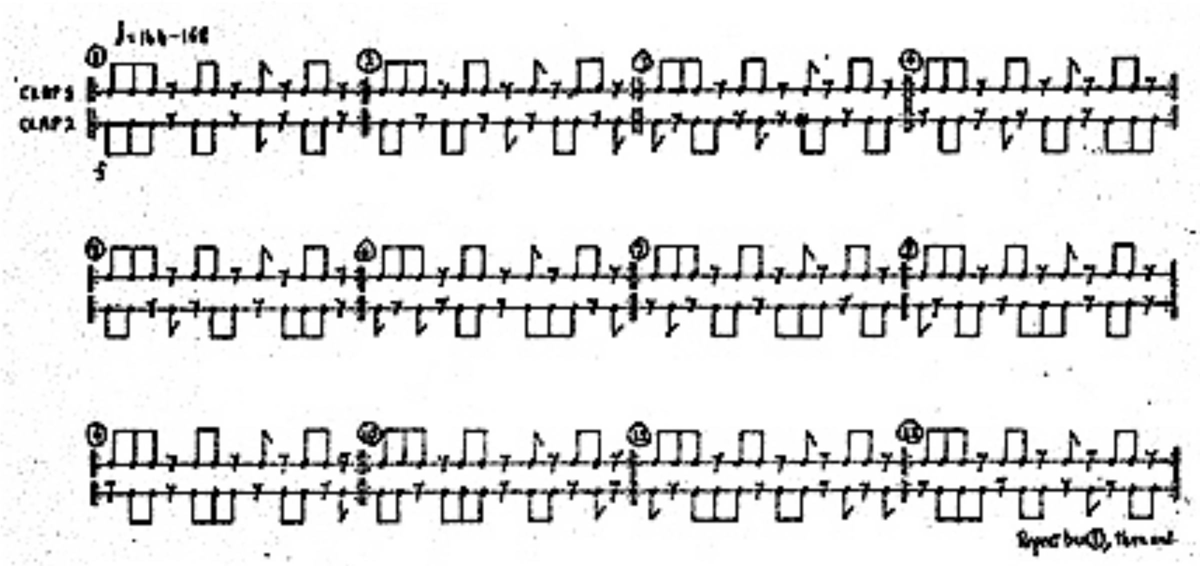
Exercises on pattern and symmetry in music

Clapping Music, Steve Reich, 1978.



Describe the structure of *Clapping Music* mathematically.

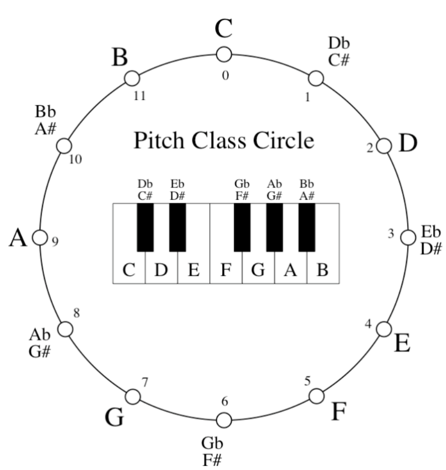
We can think of the rhythm of the first measure as the "seed" of the composition. How would using a different seed change the composition?

Why do you think Reich chose this particular seed?

Transposition, inversion, delay, and retrograde

The group of *transpositions and inversions* is the same as the symmetries of a dodecagon (musicians write T_n for rotation n places and I_n for the flip mapping 0 to n). Transpositions and inversions are often applied to chords, or sets of notes (*pitch classes*).

- For example, the D major triad {D, F#, A} is the result of transposing the set of notes of the C major triad {C, E, G} up by 2 pitches. We could write D = T_2(C).
- The A minor triad {A, C, E} is the result of inverting the C major triad so that the C note is mapped to the E note (this corresponds to f_4). We write Am = I_4(C).



The group of symmetries of an *n*-gon applies to periodic *n*-beat rhythm patterns, but in this case, the rotation operation is called *delay* and the flip is called *retrograde.*

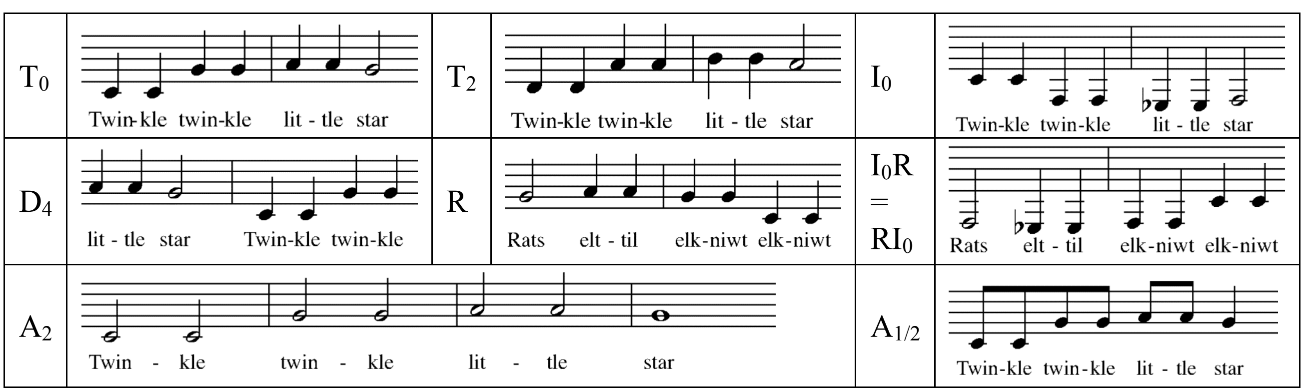
- For example, if P is the pattern in *Clapping Music,* then the second person plays P, D_{-1}(P), D_{-2}(P), …, D_{-11}(P), P; the first person just keeps playing P.
- The retrograde of the pattern `xxx.xx.x.xx.` is `.xx.x.xx.xxx` Does the retrograde pattern appear in *Clapping Music*?

Variations on a theme

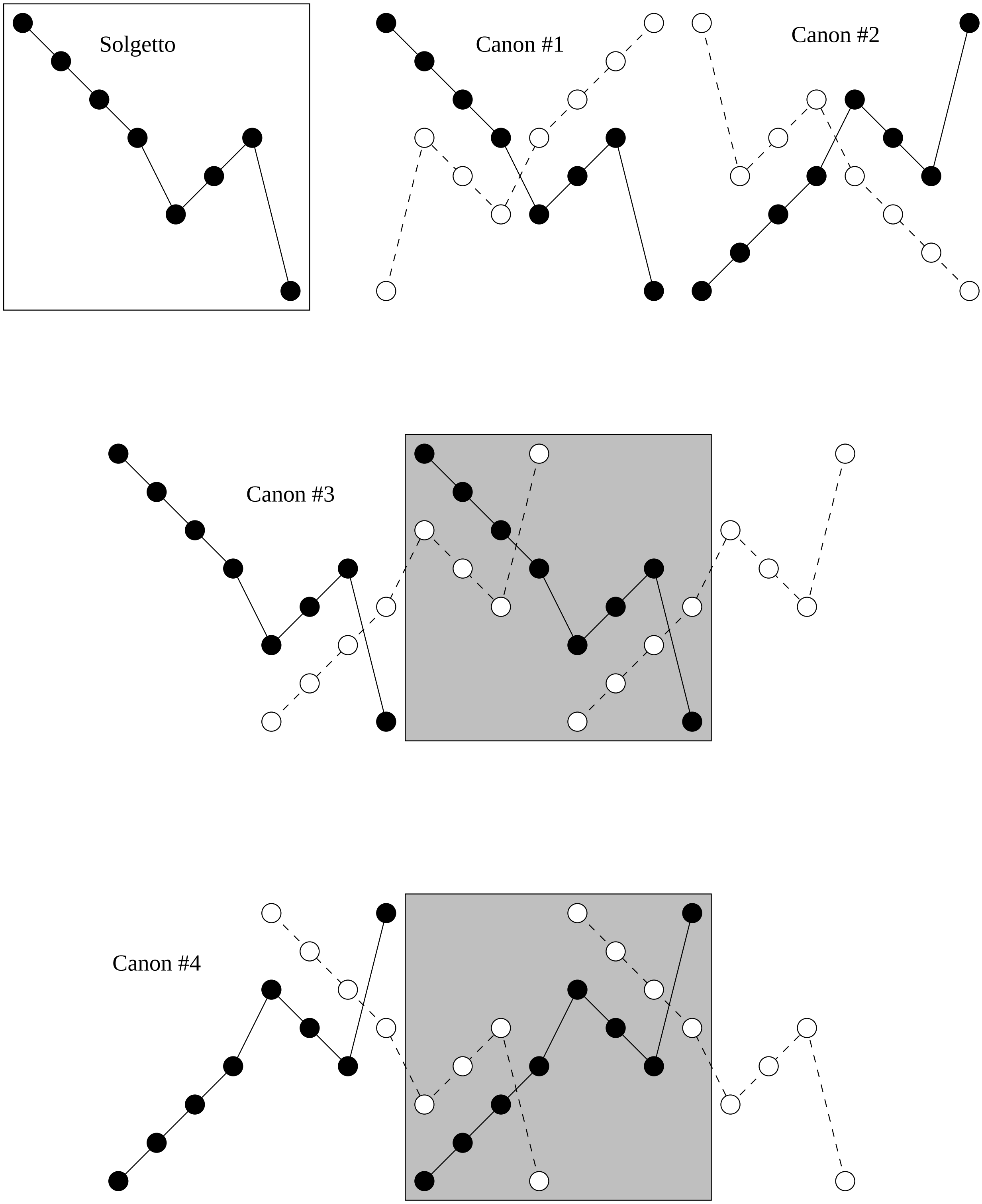
Melodies are sequences of pitches *and* rhythm patterns! The group theory becomes more complicated. You can transpose (shift the melody up or down a number of pitches), invert (turn the melody "upside down"), delay (start playing some number of beats later), reverse (apply the retrograde operation), or do any combination of these! Moreover, you can play different versions of the melody simultaneously, creating a *canon*. (There are more operations—augmentation (A) and diminution (A^{-1}), for example.) J.S. Bach used all of these operations in his "Goldberg Variations"!

Let's see how these apply to the melody "Twinkle Twinkle Little Star". First, write the melody as the sequence (0 0 7 7 9 9 7 •), then… (the "•" represents a rest)

- T_2(0 0 7 7 9 9 7 •) = (2 2 9 9 11 11 9 •)
- I_0(0 0 7 7 9 9 7 •) = (0 0 -7 -7 -9 -9 -7 •) (the "0" means that C is fixed)
- D_4(0 0 7 7 9 9 7 •) = (9 9 7 • 0 0 7 7) (this is assuming the melody repeats periodically with 8 beats per period)
- R(0 0 7 7 9 9 7 •) = (7 • 9 9 7 7 0 0)



VISUAL REPRESENTATION OF BACH'S
CANONS ON THE GOLDBERG GROUND



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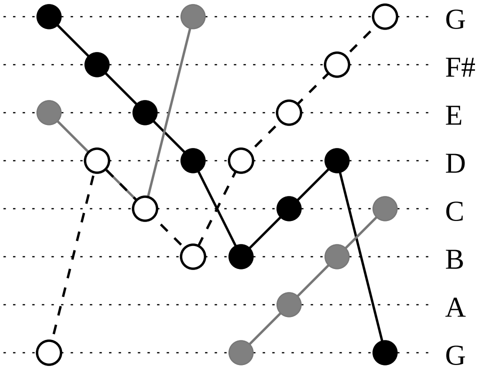
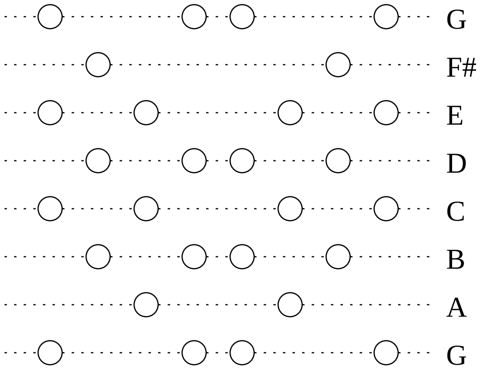


Aggregate of Bach's canons #1 and #3



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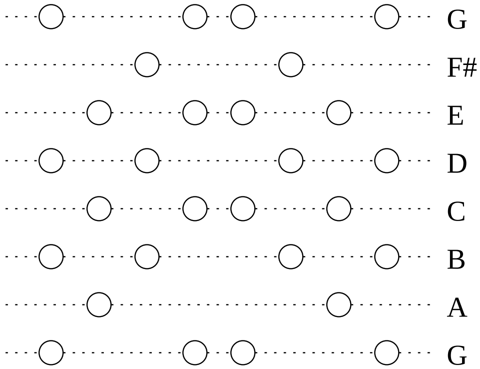

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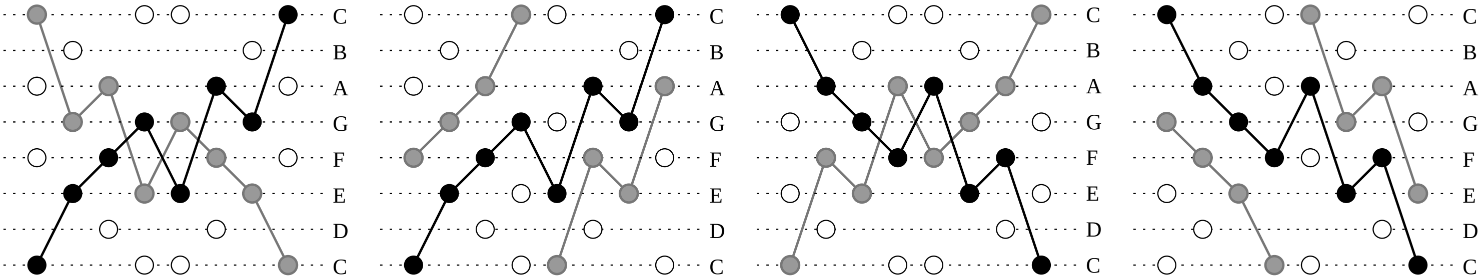

Here are four canons I wrote. Did I use the same symmetries Bach did?