Asymmetric Rhythms and Tiling Canons

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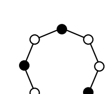
1. INTRODUCTION. If you walk into your neighborhood record store, you will be confronted with an array of different musical genres. What makes a musical style distinctive? Certainly, instrumentation is important: one does not expect to hear a trumpet playing bluegrass or a banjo in a mariachi band. However, drums and guitars are almost ubiquitous in popular music around the world; instrumentation is clearly not the whole story. Our own personal likes and dislikes are strongly influenced by the rhythms, melodic structures, harmonies, and lyrics in the songs we hear. This article focuses on what may be the most important of these aspects: rhythm. We examine the mathematics of some rhythmic structures common in popular and folk music.

Anyone who listens to rock music is familiar with the repeated drum beat—one, two, three, four—based on a 4/4 measure. Fifteen minutes listening to a Top 40 radio station offers evidence enough that most rock music has this basic beat (Audio Example 1).[1] But if we tune the radio to different frequencies, we may hear popular music (jazz, Latin, African) with different characteristic rhythms (Audio Example 2). Although much of this music is also based on the 4/4 measure, some instruments play repeated patterns that are not synchronized with the 4/4 beat, creating *syncopation*—an exciting tension between different components of the rhythm. This article is concerned with classifying and counting rhythms that are maximally syncopated in the sense that, even when shifted, they cannot be synchronized with the division of a measure into two parts. In addition, we discuss rhythms that cannot be aligned with other even divisions of the measure. Our results have a surprising application to rhythmic canons.

2. PATTERNS, CYCLES, AND ASYMMETRY. A *rhythm pattern* is a sequence of note onsets. We assume that there is an invariant unit beat that cannot be divided, so that every note onset occurs at the beginning of some beat. We identify two rhythm patterns if they have the same sequence of onsets; for example, ♩. ♩. ♩ and ♪𝄾𝄾♪𝄾𝄾♪𝄾 are equivalent. Here, we consider only periodic rhythm patterns. In this case, it is natural to deem two rhythms equivalent if one is a shift of the other; for example, 𝄆♪𝄾𝄾♪𝄾𝄾♪𝄾𝄇 is equivalent to 𝄆𝄾♪𝄾𝄾♪𝄾𝄾♪𝄇. Repeat signs (𝄆 and 𝄇) indicate periodic rhythms. We call the equivalence classes *rhythm cycles*, and sometimes call one period of the cycle a *measure*.

Here are five different notations for the same rhythm cycle (Audio Example 3):

standard	𝄆♩. ♩. ♩ 𝄇 or 𝄆♪𝄾𝄾♪𝄾𝄾♪𝄾𝄇		
drum tablature	`|: x..x..x. :|`	binary necklace	
binary	10010010		

The first line shows the standard Western musical notation. Since only note onsets, not durations, matter, we can represent the same pattern using an `x` to signify a note onset and an ellipsis point `.` to signify a rest—a notation often used by drummers. Binary

[1] Audio recordings of all examples discussed are available at `http://www.sju.edu/~rhall/Rhythms`.

notation replaces an `x` with a 1 and a `.` with a 0. An especially suggestive notation is the representation of rhythm cycles as necklaces of black and white beads, with black beads corresponding to note onsets and white ones to rests. In this case, the cyclic shift becomes a rotation. There is extensive literature on such *binary necklaces*, to which our results contribute.

Many rhythm cycles from Africa, Latin America, and Eastern Europe are *asymmetric*—that is, they cannot be broken into two parts of equal duration, where each part starts with a note onset (see [**4**, pp. 243–245]). The rhythm cycle `|: x..x..x. :|` (Audio Example 3) is asymmetric; however, the cycle `|: x.x...x. :|` is not, as it is equivalent to `|: x...x.x. :|` (Audio Example 4), which has a note onset at both the beginning and the midpoint of the measure.

Asymmetry is closely related to *syncopation*, a disruption of the expected pattern of strong and weak beats within a measure. One produces syncopation by placing an accent on a normally weak beat or not accenting a strong beat. In *duple meter*, strong beats occur at the beginning and midpoint of a measure, so asymmetric rhythm cycles are, in a sense, maximally syncopated in duple meter: although they live in a world in which measures are naturally divided in half, they cannot be delayed so that note onsets coincide with both of the strong beats within the measure. Rhythms might also be asymmetric with respect to other meters. We consider such meters in section 4.

We now translate the foregoing into mathematical terms. A rhythm pattern can be represented as a function $f : \mathbb{Z} \to \{0, 1\}$, where $f(x) = 1$ if there is a note onset on beat x and $f(x) = 0$ otherwise. If f represents a periodic rhythm of period p, it can be identified with a function with domain $\mathbb{Z}_p$. A *rhythm cycle* is defined to be an equivalence class of functions on $\mathbb{Z}_p$ modulo rotation. That is, f_1 is equivalent to f_2 if $f_1(x) = f_2(x - k)$ holds for some k and for all x. Finally, we want to consider not all periodic rhythm patterns but only those that are *asymmetric*—a notion that makes sense only if the period is even. A rhythm pattern of period $p = 2n$ is *asymmetric* if its corresponding function $f : \mathbb{Z}_{2n} \to \{0, 1\}$ does not have two onsets that are separated by half a measure (that is, $f(x) = 1$ implies that $f(x + n) = 0$).

3. COUNTING ASYMMETRIC RHYTHM CYCLES. We begin by gathering the asymmetric patterns of period $2n$ into a set S_2^n:

$$S_2^n = \{f : \mathbb{Z}_{2n} \to \{0, 1\} \ : \ f(x) = 1 \Rightarrow f(x + n) = 0\}.$$

In total, S_2^n contains 3^n functions. Indeed, if we partition the elements of $\mathbb{Z}_{2n}$ into n pairs $\{\{0, n\}, \{1, n + 1\}, \dots \{n - 1, 2n - 1\}\}$, then constructing a function f in S_2^n corresponds to choosing, for each pair, either zero or one element to be mapped to 1.

We count the total number of asymmetric rhythm cycles by starting with the set S_2^n and counting the number of equivalence classes modulo cyclic shifts (rotations). Similarly, if r is an integer such that $0 \le r \le n$ (n being the maximum possible number of onsets for an asymmetric rhythm pattern), we count the number of asymmetric rhythm cycles of period $2n$ with r note onsets by starting with the subset

$$S_2^n(r) := \{f \in S_2^n \ : \ |f^{-1}(\{1\})| = r\},$$

and counting equivalence classes modulo cyclic shifts.

In both cases, the equivalence classes are orbits induced by the group action of $\mathbb{Z}_{2n}$: for m in $\mathbb{Z}_{2n}$ and f in S_2^n or f in $S_2^n(r)$ the function $m \cdot f$ is given by $(m \cdot f)(x) = f(x - m)$. Because the equivalence classes are orbits, we can apply Burnside's lemma (for a proof, see [**5**, p. 563]).

Burnside's Lemma. *Let a finite group G act on a finite set S. For each β in G define* $\mathrm{fix}(\beta)$ *to be the number of elements s in S such that $\beta \cdot s = s$. Then the number of orbits that G induces on S is given by* $(1/|G|)\sum_{\beta\in G}\mathrm{fix}(\beta)$.

Theorems 1 and 2 originally appeared in our paper [**10**]. We omit the proofs because these theorems are the restrictions of Theorems 4 and 6 to the case $\ell = 2$.

Theorem 1. *The number of asymmetric rhythm cycles of period* $2n$ *is given by*

$$\frac{1}{2n}\left[\sum_{d|n}\phi(2d) + \sum_{\substack{d|n \\ d \text{ odd}}}\phi(d)3^{n/d}\right], \tag{1}$$

where ϕ is Euler's totient function (i.e., $\phi(d)$ is the number of integers x with $1 \le x \le d$ such that x is relatively prime to d).

Theorem 2. *If $1 \le r \le n$, then the number of asymmetric rhythm cycles with r note onsets is given by*

$$\frac{1}{2n}\sum_{\substack{d|\gcd(n,r) \\ d \text{ odd}}}\phi(d)\binom{n/d}{r/d}2^{r/d}. \tag{2}$$

4. GENERALIZATION TO ℓ-ASYMMETRY. Our characterization of asymmetry as "maximal syncopation" was based on division of the measure into two parts (duple meter). On the other hand, in section 2 we mentioned that syncopation can occur in any meter. Suppose that we divide a measure of $M = \ell n$ beats into ℓ equal parts and place a strong beat at the beginning of each part, creating "ℓ-tuple meter." We can then find rhythm cycles that are *ℓ-asymmetric*: they cannot be broken into ℓ parts of equal duration in such a way that more than one part starts with a note onset. For example, the 12-periodic rhythm `|: x.....x..x.x :|` (Audio Example 5) is 3-asymmetric ($n = 4$). No matter how we shift it, we cannot divide the measure into three equal parts so that more that one part starts with a note onset. Note that our previous definition of asymmetry corresponds to ℓ-asymmetry when $\ell = 2$.

Let n and ℓ be integers greater than two, and set $M = \ell n$. For each divisor y of M and each x in $\mathbb{Z}_M$ let $[x]_y$ denote the elements of $\mathbb{Z}_M$ that are equivalent to x modulo y. We are interested in functions $f : \mathbb{Z}_M \to \{0, 1\}$ that never assign the value 1 to two elements of any equivalence class $[x]_n$. Put

$$S_\ell^n = \left\{ f : \mathbb{Z}_M \to \{0,1\} \ : \ \sum_{y\in[x]_n} f(y) \le 1 \quad \text{for each } x \text{ in } \mathbb{Z}_M \right\}.$$

Clearly $|S_\ell^n| = (\ell+1)^n$, for any function f in S_ℓ^n is constructed by choosing from each equivalence class $[x]_n$ either zero or one element to be mapped to 1. We also examine the subset of rhythms $S_\ell^n(r)$ with r note onsets ($0 \le r \le n$):

$$S_\ell^n(r) = \{f \in S_\ell^n : |f^{-1}(\{1\})| = r\}.$$

Clearly, $|S_\ell^n(r)| = \binom{n}{r}\ell^r$. In each case, we let $\mathbb{Z}_M$ act on the set of functions and count the orbits. We use R_ℓ^n (respectively, $R_\ell^n(r)$) to denote the set of orbits of S_ℓ^n (respectively, $S_\ell^n(r)$) modulo cyclic shifts. In other words, $R_\ell^n = S_\ell^n/\mathbb{Z}_M$ and $R_\ell^n(r) = S_\ell^n(r)/\mathbb{Z}_M$.

We collect in Proposition 3 some simple observations, whose straightforward proofs we omit.

Proposition 3. *Let $dk = M = n\ell$. Then the following statements are true:*

1. $\mathrm{Gcd}(d, \ell) \cdot \mathrm{lcm}(k, n) = M$*; hence* $\gcd(d, \ell) = 1$ *if and only if* $\mathrm{lcm}(n, k) = M$.
2. *If* $\gcd(d, \ell) = 1$*, then* $n/d = k/\ell = \gcd(n, k)$.
3. *If* $\gcd(d, \ell) = 1$*, then for* x *in* $\mathbb{Z}_M$*,* $[x]_k \cap [x]_n = \{x\}$.

The total number of ℓ-asymmetric rhythm cycles.

Theorem 4. *The number of ℓ-asymmetric rhythm cycles of length $M = \ell n$ is*

$$|R_\ell^n| = \frac{1}{M}\left[\sum_{\substack{d|M \\ \gcd(d,\ell)>1}} \phi(d) \; + \sum_{\substack{d|n \\ \gcd(d,\ell)=1}} \phi(d)(\ell+1)^{n/d}\right]. \tag{3}$$

Proof. For the group $\mathbb{Z}_M$ acting on the set S_ℓ^n Burnside's lemma asserts that the number of orbits is $(1/M)\sum_{\beta\in\mathbb{Z}_M} \mathrm{fix}(\beta)$. We note first that for each divisor d of M the elements of order d are precisely the generators of the d-element subgroup $k\mathbb{Z}/M\mathbb{Z}$ of $\mathbb{Z}_M$, where $kd = M$. These are the $\phi(d)$ elements $\beta = kj$ with $1 \le j \le d$ and $\gcd(j, d) = 1$. Moreover, for each such β, $\beta \cdot f = f$ if and only if f is constant on each set $[x]_k$ (that is, f has period k).

Two cases now arise:

Case 1: $\gcd(d, \ell) > 1$. In this case $\mathrm{lcm}(k, n) < M$. Consider β in $\mathbb{Z}_M$. For $\beta \cdot f = f$ we must have $f(x) = f(x + \mathrm{lcm}(k, n))$ for each x in $\mathbb{Z}_M$. Since $x \not\equiv x + \mathrm{lcm}(k, n) \pmod M$ and since f belongs to S_ℓ^n, f must map both of these two elements to 0. We conclude that $\beta \cdot f = f$ if and only if $f(x) \equiv 0$ for all x. In this case, $\mathrm{fix}(\beta) = 1$.

Case 2: $\gcd(d, \ell) = 1$. By Proposition 3 (3), the elements of each set $[x]_k$ are pairwise incongruent modulo n, so that it is possible for a function f from S_ℓ^n to map an entire class $[x]_k$ to 1. We next need to know, given two different sets $[x_1]_k$ and $[x_2]_k$, whether there exist y_1 in $[x_1]_k$ and y_2 in $[x_2]_k$ with $y_1 \equiv y_2 \pmod n$. It is not hard to see that this occurs if and only if $x_1 \equiv x_2 \pmod{\gcd(n, k)}$. Accordingly, we partition $\mathbb{Z}_M$ by congruence modulo $k/\ell = \gcd(n, k)$ and examine the equivalence classes $[x]_{k/\ell}$, where $0 \le x \le k/\ell - 1$. Each class contains $d\ell$ elements and is in fact a union of ℓ of the sets $[x]_k$. A function $f : \mathbb{Z}_M \to \{0, 1\}$ of period k belongs to S_ℓ^n if and only if, for each class $[x]_{k/\ell}$, f does one of $\ell + 1$ things: either it maps the entire class to 0, or it maps one of the ℓ subsets $[x]_k$ contained in $[x]_{k/\ell}$ to 1 while mapping the rest of $[x]_{k/\ell}$ to 0. Constructing a function f in S_ℓ^n with the property that $\beta \cdot f = f$ thus entails making one of $\ell + 1$ choices independently for each of the classes $[x]_{k/\ell}$, so that in case 2, $\mathrm{fix}(\beta) = (\ell + 1)^{k/\ell} = (\ell + 1)^{n/d}$.

Putting the two cases together now yields the result:

$$|R_\ell^n| = \frac{1}{M}\sum_{\beta\in\mathbb{Z}_M} \mathrm{fix}(\beta) = \frac{1}{M}\left[\sum_{\substack{d|M \\ \gcd(d,\ell)>1}} \phi(d) \; + \sum_{\substack{d|n \\ \gcd(d,\ell)=1}} \phi(d)(\ell+1)^{n/d}\right].$$

Setting $\ell = 2$ in equation (3) establishes Theorem 1. ■

Corollary 5 gives us insight into the structure of asymmetric rhythms. Nonzero ℓ-asymmetric rhythms that are fixed by β of order d (> 1) are not *primitive*: they consist of d copies of some rhythm. We show that this rhythm must be ℓ-asymmetric when restricted to $\mathbb{Z}_k$. The *New Harvard Dictionary of Music* [**16**, p. 827] gives two definitions of syncopation: (1) a temporary change in the division of a measure, and (2) a contradiction of the normal placement of strong and weak beats. Our definition of ℓ-asymmetry follows (2). However, ℓ-asymmetric rhythms fixed by an element of order d (> 1) are also syncopated in the sense of definition (1): the measure is divided into d parts, where d is relatively prime to ℓ, producing the effect musicians call "d against ℓ." When $d = 3$ and $\ell = 2$, the effect is also called a *hemiola*—a triplet in duple meter.

Fix any divisor d of M, and let β be an element of $\mathbb{Z}_M$ of order d. If $f : \mathbb{Z}_M \to \{0, 1\}$ is a function fixed by β, then f is periodic with period $k = M/d$. Let $\tilde{f} : \mathbb{Z}_k \to \{0, 1\}$ be the natural restriction of f to $\mathbb{Z}_k$—that is, for $x = 0, \dots, k - 1$, let $\tilde{f}(x) = f(x)$.

Corollary 5. *Let f, $\tilde{f}$, d, and k be as indicated. The following statements hold:*

1. *If f is not identically zero, then* $\gcd(d, \ell) = 1$.
2. *A function f is in S_ℓ^n if and only if $\tilde{f}$ is in $S_\ell^{k/\ell}$.*

Proof. Statement (1) is exactly what is proved in case 1 of Theorem 4. Turning to the second assertion, assume that f is fixed by β, so f is determined by its values on $K = \{0, 1, \dots, k - 1\} \subseteq \mathbb{Z}_M$. If $\gcd(d, \ell) > 1$, then f and $\tilde{f}$ are both identically zero, placing f in S_ℓ^n and $\tilde{f}$ in $S_\ell^{k/\ell}$. Now let $\gcd(d, \ell) = 1$. As shown in the proof of Theorem 4, f is in S_ℓ^n if and only if, for each class $[x]_{k/\ell}$, either f maps the entire class to 0, or f maps exactly one subset $[x]_k$ of $[x]_{k/\ell}$ to 1 while mapping the rest of $[x]_{k/\ell}$ to 0. Now K is a complete set of residues modulo k, so f is in S_ℓ^n if and only if, when $0 \le x \le k/\ell - 1$, either f maps all of the set $[x]_{k/\ell} \cap K$ to 0, or f maps exactly one element of this set to 1. But if one identifies K with $\mathbb{Z}_k$, this becomes precisely the condition $\tilde{f}$ must satisfy to be in $S_\ell^{k/\ell}$. ■

Thus, when $\gcd(d, \ell) = 1$, elements of S_ℓ^n that are fixed by β of order d are in bijection with elements of $S_\ell^{k/\ell} = S_\ell^{n/d}$. Since $|S_\ell^{n/d}| = (\ell + 1)^{n/d}$, Corollary 5 gives a second perspective on the equation $\mathrm{fix}(\beta) = (\ell + 1)^{n/d}$.

Example. Let $\ell = 5$ and $n = 6$. Corollary 5 tells us that any 5-asymmetric rhythm that is fixed by an element of order $d = 3$ consists of three copies of a rhythm that is 5-asymmetric modulo $k = 10$. Audio Example 6 is generated by the 5-asymmetric string `x..x......`; divisions of the measure into five parts are shown:

```
|x..x..|....x.|.x....|..x..x|......|
```

A musician would think of this rhythm as "three against five," because it involves the division of a measure in quintuple meter into three parts. This type of asymmetry is common in Central African music [**4**, p. 245].

The number of r-note ℓ-asymmetric rhythm cycles. Now consider $S_\ell^n(r)$. Recall that $R_\ell^n(r) = S_\ell^n(r)/\mathbb{Z}_M$. Clearly, $|R_\ell^n(0)| = 1$.

Theorem 6. *If $1 \le r \le n$, then the number of ℓ-asymmetric rhythm cycles of length $M = \ell n$ with r onsets is given by*

$$|R_\ell^n(r)| = \frac{1}{M} \sum_{\substack{d \mid \gcd(n,r) \\ \gcd(d,\ell)=1}} \phi(d) \binom{n/d}{r/d} \ell^{r/d}. \tag{4}$$

Proof. In outline, the proof is similar to that of Theorem 4. The only difference is that one must consider three cases, not two: $\gcd(d, \ell) > 1$; $\gcd(d, \ell) = 1$ and d does not divide r; and $\gcd(d, \ell) = 1$ and d divides r. Setting $\ell = 2$ in equation (4) establishes Theorem 2. ■

Example. We list the rhythm cycles of length twelve that are 3-asymmetric and have four note onsets. Using our formula for the number of r-note rhythm cycles, where $\ell = 3$ and $r = n = 4$, we see that there must be eight such cycles, which we list as follows (Audio Example 7):

```
1. |: xxxx........ :|        5. |: xx.x..x..... :|
2. |: xxx....x.... :|        6. |: x.....x..x.x :|
3. |: xx....xx.... :|        7. |: x.x..x.x.... :|
4. |: xx.x......x. :|        8. |: x..x..x..x.. :|
```

Notice that cycles 3 and 8 are not primitive. Cycles 5 and 6 are inversions of each other (that is, cycle 5 is cycle 6 played backwards); each of the other cycles is its own inversion.

Primitive ℓ-asymmetric rhythm cycles. Corollary 5 suggests that we also direct our attention to primitive rhythm cycles. A rhythm pattern of period M is *primitive* if no element β of $\mathbb{Z}_M$ other than the identity fixes it. Similarly, we deem a rhythm cycle primitive if the patterns in it are primitive. We wish to count primitive ℓ-asymmetric rhythm cycles. Let P_ℓ^n denote the set of primitive cycles in R_ℓ^n. Observe that each cycle in P_ℓ^n contains exactly M patterns, namely, the M distinct cyclic shifts of any pattern in the cycle. Thus, the number of ℓ-asymmetric rhythm patterns of period M is given by $M \cdot |P_\ell^n|$. In what follows μ signifies the classical Möbius function (i.e., $\mu(1) = 1$, $\mu(d) = 0$ if d is divisible by the square of any prime, $\mu(d) = 1$ if d is the product of an even number of distinct primes, and $\mu(d) = -1$ if d is the product of an odd number of distinct primes).

Theorem 7. *If n, ℓ, and M are as described, then*

$$|P_\ell^n| = \frac{1}{M} \sum_{\substack{d \mid n \\ \gcd(d,\ell)=1}} \mu(d)\big[(\ell+1)^{n/d} - 1\big]. \tag{5}$$

Proof. Let x be the product (counting multiplicities) of all primes that divide n but do not divide ℓ; in other words, if $\ell = q_1^{a_1} \cdots q_s^{a_s}$ and $n = q_1^{b_1} \cdots q_s^{b_s} p_1^{c_1} \cdots p_t^{c_t}$, where the p_i and q_j are distinct primes and the a_i and c_j are all positive, then $x = p_1^{c_1} \cdots p_t^{c_t}$ (if there are no p_i, then $x = 1$). Observe that $d|x$ if and only if $d|n$ and $\gcd(d, \ell) = 1$. Now consider S_ℓ^n. By Corollary 5, each nonzero string f in S_ℓ^n comprises d copies of a primitive string in f_d in $S_\ell^{n/d}$ for a unique divisor d of x and a unique primitive function f_d, so

$$(\ell+1)^n - 1 = \sum_{d|x} \frac{M}{d} \left| P_\ell^{n/d} \right|.$$

Furthermore, applying the same argument to $S_\ell^{n/d}$, we see that

$$(\ell+1)^{n/d} - 1 = \sum_{d'|d} \frac{M/d}{d'} \left| P_\ell^{(n/d)/d'} \right|$$

for each divisor d of x. By Möbius inversion over the divisors of x (see [**11**, p. 236]),

$$M \cdot |P_\ell^n| = \sum_{d|x} \mu(d)\Big[(\ell+1)^{n/d} - 1\Big] = \sum_{\substack{d|n \\ \gcd(d,\ell)=1}} \mu(d)\Big[(\ell+1)^{n/d} - 1\Big]. \qquad \blacksquare$$

Observe that when $x > 1$, formula (5) reduces to

$$|P_\ell^n| = \frac{1}{M}\Bigg[\sum_{\substack{d|n \\ \gcd(d,\ell)=1}} \mu(d)(\ell+1)^{n/d}\Bigg], \tag{6}$$

because $\sum_{d|x} \mu(d) = 0$ in this case.

These methods also allow us to count $P_\ell^n(r)$, the set of r-note cycles in P_ℓ^n:

Theorem 8. *If $1 \le r \le n$, then the number of primitive ℓ-asymmetric rhythm cycles of length $M = \ell n$ with r onsets is given by*

$$|P_\ell^n(r)| = \frac{1}{M} \sum_{\substack{d|\gcd(n,r) \\ \gcd(d,\ell)=1}} \mu(d) \binom{n/d}{r/d} \ell^{r/d}. \tag{7}$$

Self-complementary asymmetric rhythm cycles. In general, the *complement* f^c of a rhythm pattern f is formed by exchanging note onsets and rests; that is, $f^c = 1 - f$. A pattern of period p is *self-complementary* if f and f^c are in the same rhythm cycle. A 2-asymmetric rhythm pattern f of period $2n$ is self-complementary if and only if $|f^{-1}(\{1\})| = n$. Thus, $|R_2^n(n)|$ gives the number of self-complementary asymmetric cycles, furnishing an alternate proof of [**9**, p. 172] (reported in [**17**]), which is the solution to a problem easily seen to be equivalent to this one. It is sequence A000016 in [**18**]. We note that $|P_2^n(n)|$, which is closely related to $|R_2^n(n)|$, is sequence A000048 in [**18**].

5. RHYTHMIC CANONS.

Our results on ℓ-asymmetry have surprising applications to rhythmic canons. A *canon* is a musical figure produced when two or more voices play the same melody, with each voice starting at a different time. Simple canons are called *rounds* (for example, “Frère Jacques” and “Row, Row, Row Your Boat”). We consider canons in which rhythms, and not necessarily melodies, are duplicated by each voice. A *rhythmic canon* is produced when each voice plays a rhythm pattern, called the *inner rhythm*, and the voices are offset by amounts determined by a second pattern called the *outer rhythm*. We assume that both inner and outer rhythms can be expressed in terms of the same unit beat. Rhythmic canons were first studied mathematically by Vuza [**19**], [**21**], [**20**], [**22**] (see [**2**], [**3**], [**1**], or [**7**] for further background).

The composer Olivier Messiaen (1908–1992), who coined the term "rhythmic canon," used rhythmic canons in his work (*Harawi*, "Adieu," and *Visions de l'Amen*, "Amen des anges, des saints, du chant des oiseaux"). He describes the sound of a rhythmic canon as a sort of "organized chaos" [**13**, p. 46]. In both pieces, he uses the inner rhythm

```
x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
```

together with the outer rhythm `e.e.e` (`e` indicates some entry) to generate the canon heard in Audio Example 8:

```
Voice 1:  x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
Voice 2:    x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
Voice 3:      x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
                                      *                * * * *          *
Entries:  e.e.e
```

In this notation, vertical alignment corresponds to alignment in time. A rhythmic canon is called *complementary* if, on each beat, no more than one voice has a note onset. For the most part, Messiaen's canon is complementary; asterisks mark deviations from this rule.

Recent work on rhythmic canons ([**19**], [**21**], [**20**], [**22**], [**2**], [**3**], [**1**], [**7**]) concerns canons of periodic rhythms. For example, Audio Example 9 is a 12-beat complementary canon generated by the inner rhythm `|: x.....x..x.x :|` and outer rhythm `|: e...e...e... :|`:

```
Voice 1:  x . . . . . x . . x . x |: x . . . . . x . . x . x :|
Voice 2:          x . . . . . x . |: . x . x x . . . . . x . :|
Voice 3:                  x . . . |: . . x . . x . x x . . . :|

Entries:  e . . . e . . . e . . . |: e . . . e . . . e . . . :|
```

The actual canon is contained between the repeat signs; the pattern to the left is included to show how the voice entries are staggered. The canon is complementary, because for each x in $\mathbb{Z}_{12}$ beats x, $x+4$, and $x+8$ contain no more than one onset. In other words, the inner rhythm is a 3-asymmetric rhythm cycle of length twelve. In general, a periodic rhythmic canon of ℓ voices, each spaced by n notes from the previous, is complementary if and only if its inner rhythm is ℓ-asymmetric.

A *tiling canon* is a canon of periodic rhythms that has *exactly* one note onset (in some voice) per unit beat. Rhythmic tiling canons were introduced as *regular complementary canons* by Vuza [**19**], [**21**], [**20**], [**22**] (see also Andreatta et al. [**2**], [**3**]). The example just discussed is a tiling canon. In general, a rhythm of period $\ell n = M$ forms a tiling canon of ℓ equally-spaced voices if and only if it is ℓ-asymmetric and has n note onsets.

Our results enumerate complementary and tiling canons in which both the inner and outer rhythms are periodic and in which the outer rhythm has equally-spaced onsets. In particular, the number of M-periodic complementary canons of ℓ equally-spaced voices is $|R_\ell^n|$; the number of such canons in which the inner rhythm has r notes is $|R_\ell^n(r)|$; and the number of M-periodic tiling canons of ℓ equally-spaced voices is $|R_\ell^n(n)|$. Our formulas for $|P_\ell^n|$, $|P_\ell^n(r)|$, and $|P_\ell^n(n)|$ give analogous results for primitive canons.

In section 4, we listed the eight 3-asymmetric cycles of length twelve with four onsets. Each of these determines a tiling canon (Audio Example 10). It is interesting to hear how the degree of symmetry affects the sound of the resulting canon: patterns 5 and 6 sound to us the "most syncopated."

6. OPEN PROBLEMS. Our condition that the voices in a canon be equally spaced greatly simplifies the problem of enumerating them. There exist complementary and tiling canons with unequally-spaced voices: for example, the inner rhythm `|: x.x.....  :|` and outer rhythm `|: ee..ee.. :|` define a tiling canon (Audio Example 11). Fripertinger [**8**] has computed the numbers of tiling canons up to period forty. The question of finding analogues of our results for general rhythmic canons is open.

An extreme example occurs when both the inner and outer rhythms of a tiling canon are primitive, producing a *tiling canon of maximal category*. Vuza showed that no nontrivial tiling canons of maximal category exist for periods less than seventy-two [**19**, Theorem 2.2, p. 33]. Audio Example 12 is a six-voice canon of maximal category and period seventy-two. There is no known formula for the number of tiling canons of maximal category.

As noted earlier, the *inversion* of a rhythm pattern is that pattern played backwards. Beethoven uses a modified tiling canon in which the rhythm patterns are inversions of each other in his string quartet Op. 59, no. 2, in which the patterns are `xx..x.` and `..xx.x`. The problem of finding tiling canons using one rhythm and its inversion is equivalent to one studied by Meyerowitz [**14**]. His result implies that any rhythm with three onsets must tile in this way, but the general question remains open. We have analogues of Theorems 4 and 6 for the case in which the full dihedral group acts on S_ℓ^n and $S_\ell^n(r)$, and we have posted these results on our website.

A rhythm is *augmented* by multiplying all of its note durations by a constant value. For example, `x.x.x...` and `x..x..x.....` are both augmentations of `xxx.` (in the first case note durations are multiplied by two, and in the second case by three). The composer Tom Johnson (b. 1939) has used tiling canons of rhythms in which augmentation is allowed; an example appears in [**12**]. Johnson and others [**3**] have posed several problems concerning canons with augmentation, some of which remain open.

Rhythmic tiling canons are, in fact, one-dimensional tilings of the integers using a single tile. All such tilings are periodic [**15**]. Our results on tiling canons give the number of tilings of $\mathbb{Z}$ by equally-spaced placements of a single tile. There are many open questions on one-dimensional tilings (for background, see [**6**] or [**7**]).

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