Math for Poets and Drummers

The Mathematics of Rhythm

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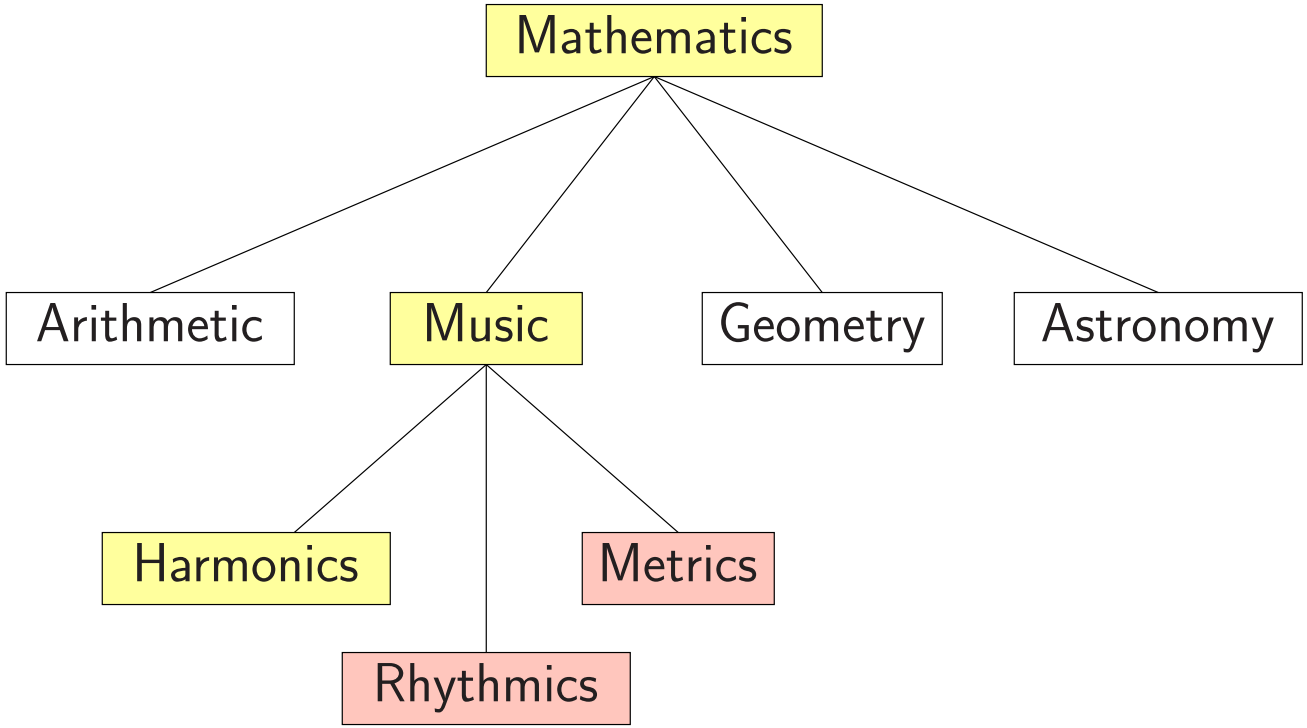
Listen

Cassiodorus (6th century)



Meter in Poetry

Much poetry follows one of a set of rhythmic rules, called *meters*.

In English, the meter of a poem is determined by its pattern of stressed and unstressed syllables.

In Sanskrit (and many other languages), meter is determined by the pattern of long and short syllables. There are dozens of meters; some are determined by the number of syllables in a line, and some by the total duration of the syllables in a line.

Names of meters and numbers of syllables

ukta	1	*pratistha*	4	*usnik*	7	*pankti*	10
atyukta	2	*supratistha*	5	*anustubh*	8	*tristubh*	11
madhya	3	*gayatri*	6	*brhati*	9	...	...

Pingala's Chandahsutra (c. 200 B.C.)

Syllables are short or long; in duration

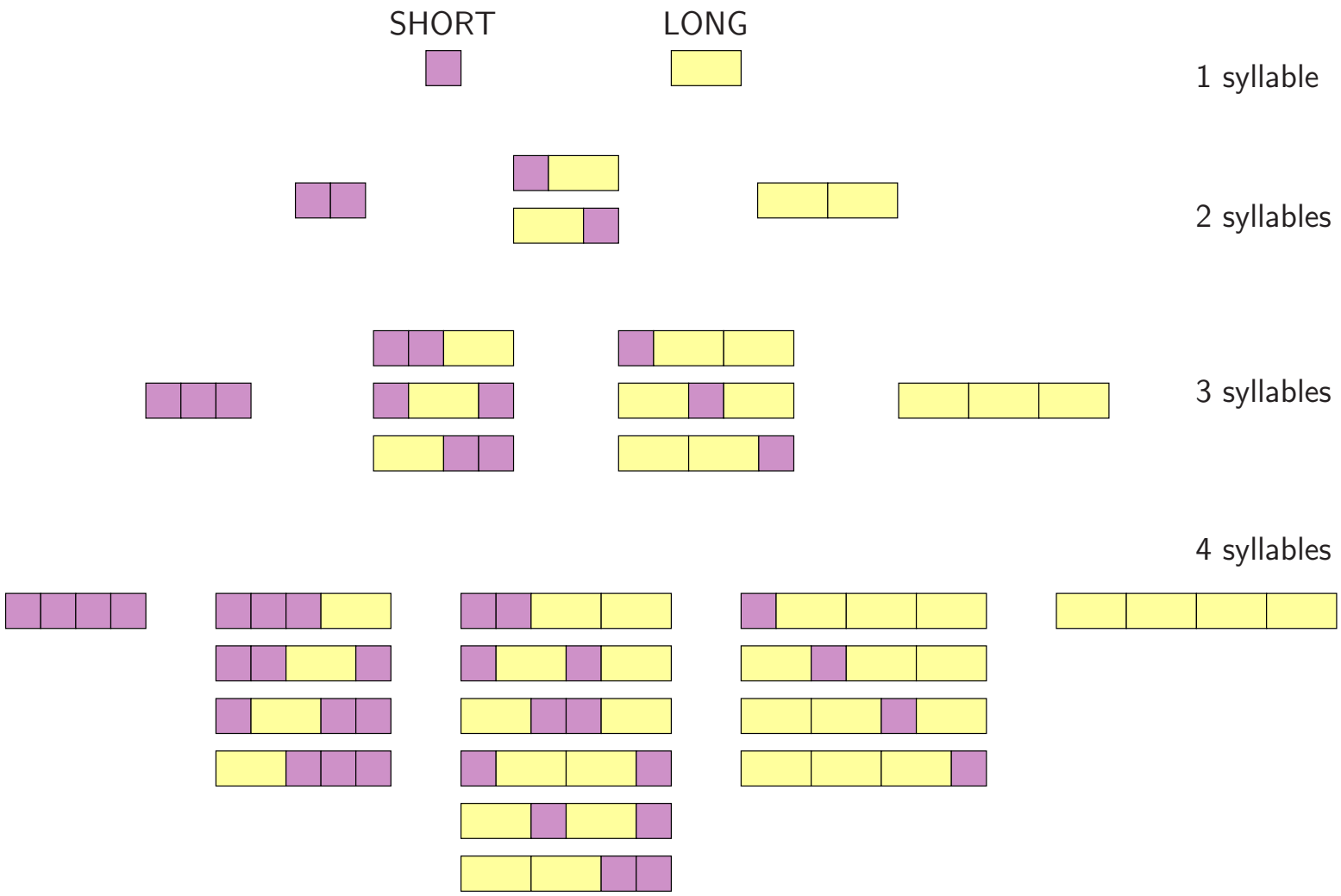
$$1 \text{ long} = 2 \text{ short}$$

Pingala [1] classified the 16 different meters of four syllables like this:

1	meter of four short syllables	SSSS
4	meters of three shorts and a long	SSSL, SSLS, SLSS, LSSS
6	meters of two shorts and two longs	LLSS, LSSL, SSLL, SLLS, LSLS, SLSL
4	meters of one short and three longs	SLLL, LSLL, LLSL, LLLS
1	meter of four longs	LLLL

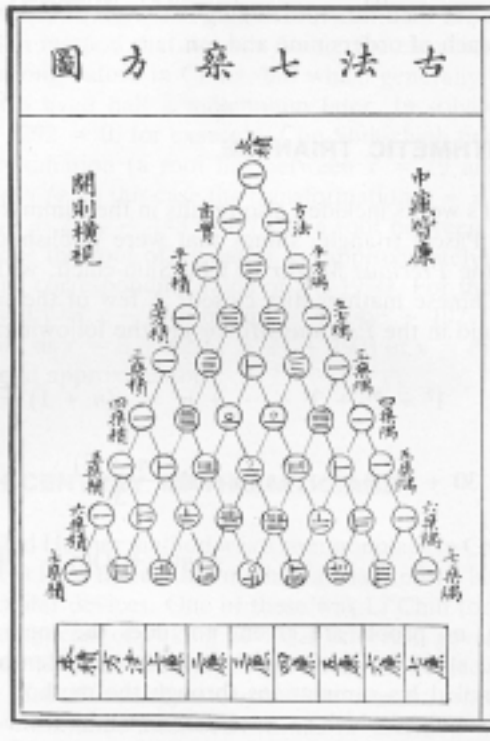
He described a very interesting pattern...

The Pattern. . .



“Pascal’s” Triangle

North Africa, c. 1150
As Samaw’al ibn Yahya
al-Maghribi



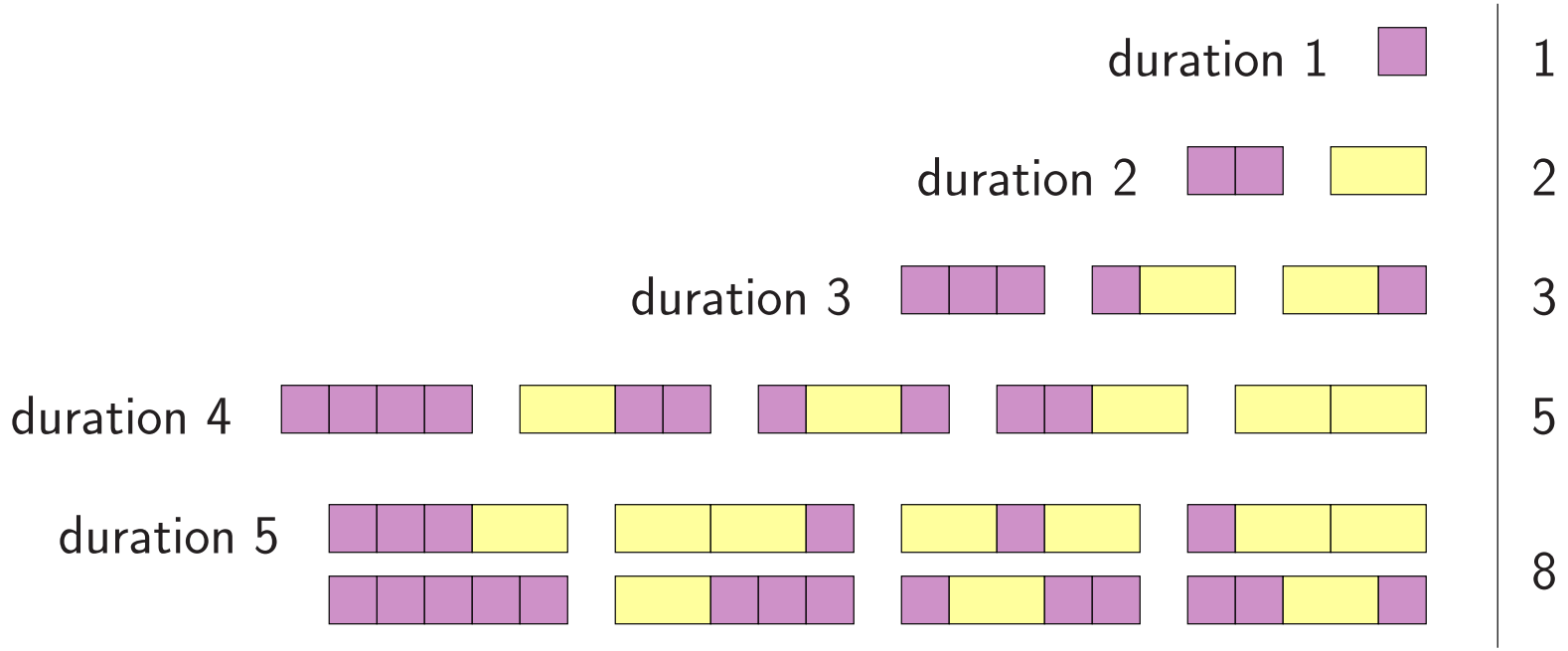
China, 1303
Chu Shih-Chieh

Germany, 1527
Petrus Apianus

Ācārya Hemacandra (c. 1150 AD)

Hemacandra enumerated meters in which the duration of a line is fixed, but the number of syllables is not [3].

Here are the possible meters of duration 1 to 5. What's the pattern? Make a conjecture!



The "Fibonacci" Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ...

The first Fibonacci number is 1. After that, each number is the sum of the two previous numbers. A mathematician would express this relationship as

$$F_N = F_{N-1} + F_{N-2}$$

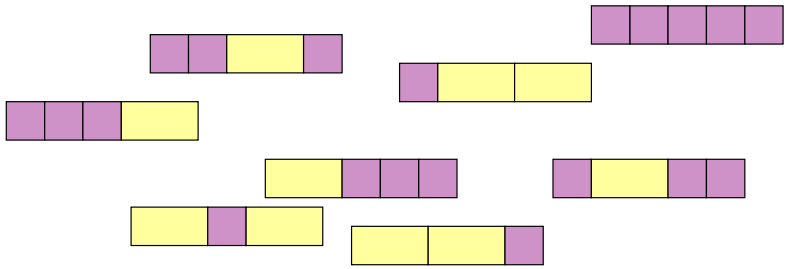
where F_N is the Nth Fibonacci number.

These numbers have an *amazing* variety of applications, not only to poetry, but also to nature, art, music, science, and many areas of mathematics.

`http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`

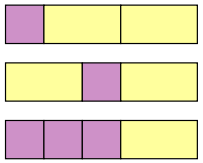
Got Proof?

Start with the patterns of duration N in one big pile

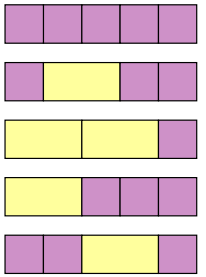


Separate them into two smaller piles:

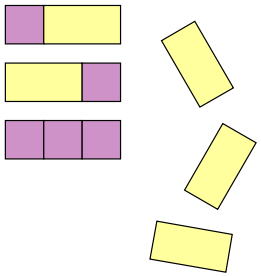
those ending in a long

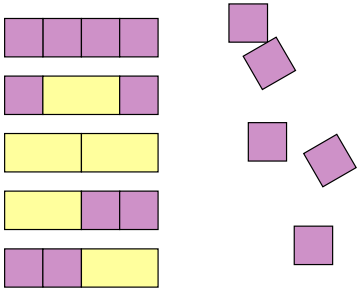


those ending in a short



OFF WITH the last syllables!





Drum Roll, Please. . .

There is 1 meter of duration 1.
There are 2 meters of duration 2.

For $N = 3, 4, 5, \ldots$, the number of meters of duration N is equal to

(number of meters of duration $N - 2$) + (number of meters of duration $N - 1$).

This is precisely the Fibonacci relationship!

Musical Rhythm

Rhythms in music are patterns of note onsets or accents. A *note* is the interval between successive attacks. Its poetic analog is a syllable.

Notation

Here are different ways to write the same rhythm pattern. Listen

standard	 or
additive	3 + 3 + 2
drum tablature	`x..x..x.`

There are two types of rhythm: *additive* and *divisive*. Most music combines elements of both.

Additive Rhythm

Additive rhythm has an underlying unit pulse. Rhythm patterns are formed by combining unit pulses into notes. Plainchant, Indian music, and Eastern European music are primarily additive. The rhythm of Sanskrit poetry is also additive.

Examples: (1) Medieval polyphony (2) Turkish dance, Okay Temiz Listen

Challenge

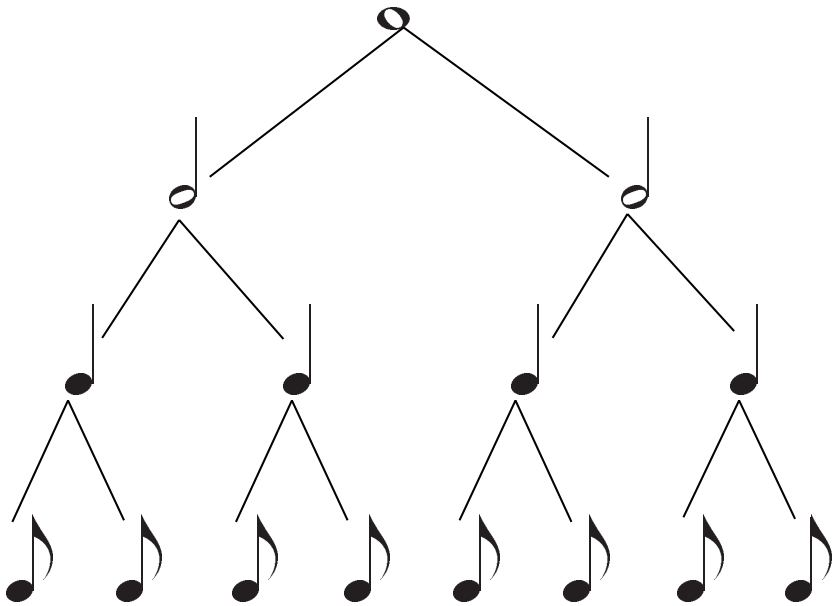
In Eastern European music, rhythms are composed of long and short notes, where a long note is 3 unit pulses and a short note is 2 unit pulses. For example, the Bulgarian *lesnoto* rhythm is a 7-beat rhythm of a long note followed by 2 short notes, written $3+2+2$, while the *rachenitsa* rhythm is $2+2+3$.

What is the formula for the number of additive rhythms of length N?

Divisive Rhythm

Divisive rhythm occurs when the basic unit of time (a measure) is subdivided into a number of equal notes, each of which may be further subdivided. Western musical notation emphasizes this view of rhythm.

Here's the classic picture. In this case, all divisions are binary. Listen



Division Type

We will classify divisive rhythms as having *division type* $(a_1, a_2, \ldots, a_d)$, meaning that the rhythm is organized by dividing a measure into a_1 equal parts, then dividing each part into a_2 parts, and so on. We will use exponential notation when there can be no confusion.

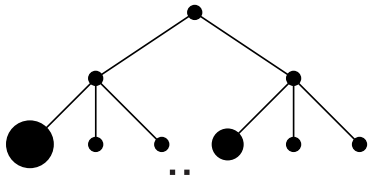
As we continue to divide, the accents on newly created beats get weaker.

Examples.

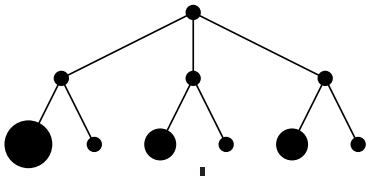
- $(2, 2, 2, 2, 2) = 2^5$ (an example of binary division) Listen
- type $(2, 3)$ (a jig—6/8 time) Listen
- type $(3, 2)$ (a waltz—3/4 time) Listen

Additive vs. Divisive

Divisive rhythm has an underlying structure that gives each beat a natural stress. The hierarchy of beats is determined by the division type. Here, size indicates accent.



jig



waltz

However, in additive rhythm each beat has equal natural stress. Additive rhythm has no underlying organization of the measure other than the pulse.

Syncopation

The rhythm  $= 3 + 3 + 2$ is an additive pattern of three notes and duration 8 pulses. It creates no tension in additive meter.

However, the same set of note onsets is perceived as a deviation from the expected pattern of accents in the division type 2^3. This deviation creates a rhythmic tension known as *syncopation*.

Syncopated rhythm is common in African-American and Latin American music. It is also highly developed in Renaissance music.

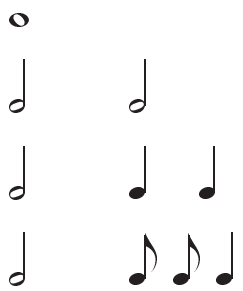
Examples: (1) E. T. Mensah, Ghanaian pop song Listen
(2) Cuarteto Patria, "Chan-chan" Listen

If the division type is binary, how many syncopated rhythm patterns of k notes are there?

Uniform Rhythm Patterns

Uniform rhythms are definitely **not** syncopated.
Example: (1) The Beatles, "Free as a Bird" Listen (2) MIDI Listen

Let a *uniform rhythm* be any rhythm pattern that results from a sequence of equal divisions of notes. For example,  is a uniform rhythm of division type 2^3. The corresponding sequence would be



However,  is not uniform, since the first half of the measure has been divided in the ratio $3:1$.

The Catalan numbers got rhythm!

Theorem 1 (Hall and Klingsberg) *The number of uniform rhythm patterns of binary division type with k notes equals the Catalan number $C(k-1) = (2k-2)!/(k!(k-1)!)$.*

Catalan numbers: 1, 1, 2, 5, 14, 42, 132, 429, 1430,. . .

Are "Non-uniform" and "Syncopated" Equivalent?

All uniform rhythms are non-syncopated. However, a musician would probably not agree that all non-syncopated rhythms are uniform. For example,  does not sound syncopated.

According to the *New Harvard Dictionary of Music*, syncopation is "... a momentary contradiction of the prevailing meter or pulse. This may take the form of a temporary transformation of the fundamental character of the meter ... or it may be simply the contradiction of the regular succession of strong and weak beats within a measure or group of measures [2]."

So  is not syncopated because long notes correspond to strong beats in division type 2^3.

Asymmetric Rhythms

Paul Klingsberg and I have studied asymmetric rhythms, which are commonly found in African drumming. They are rhythms that are "maximally syncopated" in that, even when delayed, they never are aligned with the primary divisions in a measure.

Examples: (1) Aka Pygmies, "Bobangi" Listen (2) MIDI Listen

Our results had surprising applications to the study of rhythmic canons. . .

Listen

Listen

For more information, see
`http://www.sju.edu/~rhall/research.html`

References

[1] Amulya Kumar Bag. Binomial theorem in ancient India. *Indian J. History Sci.*, 1:68–74, 1966.

[2] Don Randel, editor. *The new Harvard dictionary of music*. Belknap Press, Cambridge, MA, 1986.

[3] Parmanand Singh. Ācāraya Hemacandra and the (so-called) Fibonacci numbers. *Math. Ed. (Siwan)*, 20(1):28–30, 1986.