Math for Poets (and Drummers)

Rachel Hall • Saint Joseph's University
rhall@sju.edu • www.sju.edu/∼rhall
January 10th, 2004

Inspiration

- Historical discoveries of Pascal's Triangle and the Fibonacci numbers through study of poetic meter.
- Musical notation and fractions: time signature (4/4, 9/8, etc.); note values—whole, half, and quarter notes.
- Combinatorics of Indian drumming exercises. For example, *takita takita taki ; takita taki takita ; taki takita takita*.
- Love of folk and world music.

Historical Sources

Pingala. Pingala's *Chandahsutra* (c. 200 B.C.) enumerated the possible poetic meters of a fixed number of syllables [1]. Syllables are short or long.

For example, he classified the 16 different meters of four syllables in the following way: There are 1 meter of four short syllables (SSSS); 4 meters of three shorts and a long ; 6 meters of two shorts and two longs ; 4 meters of one short and three longs ; 1 meter of four longs.

He proceeds to describe the pattern—Pascal's Triangle!—for classifying meters of any given number of syllables.

Hemacandra. The Jain writer Ācārya Hemacandra (c. 1150 AD) set the duration of a long syllable to be twice that of a short syllable (L = 2S). He enumerated meters in which the duration of a line is fixed, but the number of syllables is not [3].

Let's list the possible patterns.

duration	patterns	number
1	S	1
2	SS, L	2
3	SL, SSS, LS	3
4	SSL, LL, SLS, SSSS, LSS	5

Thus, Hemacandra discovered the Fibonacci numbers!

Rhythm

Rhythm is a pattern of events in time. Rhythm is found in nature, music, poetry, speech, and dance. Two definitions specific to music and poetry:

- A rhythm is a pattern of note onsets, syllables, or accents. We will also refer to *rhythm patterns* or *rhythms*.
- Rhythm is overall temporal organization of a musical piece or poem. For example, time signature is a component of rhythm in this sense.

Types of Rhythm

There are two types of rhythm: *additive* and *divisive*. Most music combines elements of both.

Additive Rhythm

Additive rhythm (also called durational rhythm) has an underlying *unit pulse*. Rhythm patterns are formed by combining unit pulses into notes. Plainchant, Indian, and Eastern European music is primarily additive. A *note* is the interval between successive attacks. Its poetic analog is a syllable. The *duration* or *length* of an additive rhythm pattern is the number of unit pulses it occupies.

Examples. (1) Medieval polyphony[1] (2) Bulgarian dance [2]

Notation

Here are different ways to notate the same rhythm pattern. Each notation is suggestive of a different way of studying additive rhythm.

standard	
additive	$3 + 3 + 2$
drum tablature	`x..x..x.`
binary	10010010

Since we have defined rhythm patterns to be determined only by attacks or accents, there are multiple ways to represent the same pattern in standard notation.

Rhythm Cycles

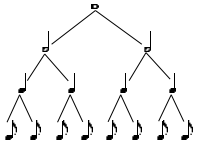
Much drumming consists of repeated patterns, which we will call rhythm cycles. Two rhythm cycles are equivalent if one results from shifting the other by an integer number of unit pulses. The following rhythm cycles form an equivalence class: `x.xx` `.xxx` `xxx.` `xx.x`

Binary Necklaces

Rhythm cycles of duration n correspond to *binary necklaces* of n beads, either black or white. Note onsets correspond to black beads. The above class of rhythms corresponds to this necklace:  Some results about binary necklaces have interesting musical interpretations.

Divisive Rhythm

Divisive rhythm occurs when the basic unit of time (usually a measure) is subdivided into a number of equal notes, each of which may be further subdivided. Western musical notation emphasizes this view of rhythm. Here's the classic picture. In this case, all divisions are binary.



Rhythm patterns correspond to subsets of beats at the lowest level.

Division Type

We will classify divisive rhythms as having *division type* $(a_1, a_2, \ldots, a_d)$, meaning that the rhythm is organized by

[1] Anonymous 4, Kyrie christifera, *An English Ladymass*. Harmonia Mundi, HMU 907080, 1992.

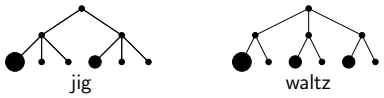
[2] Troika, Yambolska Ratchenitsa, *Balkana*. Hannibal, HNCD 1335, 1987.

dividing a measure into a_1 equal parts, then dividing each part into a_2 parts, and so on. We will use exponential notation when there can be no confusion. As we continue to divide, the accents on newly created beats get weaker.

Examples. (1) $(2,2,2,2,2) = 2^5$ (2) type $(2,3)$ (a jig—6/8 time) (3) type $(3,2)$ (a waltz—3/4 time)

Additive vs. Divisive

Divisive rhythm has an underlying structure that gives each beat a natural stress. The hierarchy of beats is determined by the division type. Here, size indicates accent.



However, in additive rhythm each beat has equal natural stress. Additive rhythm has no underlying organization of the measure other than the pulse.

An additive rhythm pattern cannot start with a rest, nor is the *empty rhythm* counted as an additive rhythm. However, we will include rhythm patterns starting with rests and the empty rhythm as divisive rhythms.

Syncopation

The rhythm  $= 3 + 3 + 2$ is an additive pattern of three notes and duration 8 pulses. It creates no tension in additive meter.

However, the same set of note onsets is perceived as a deviation from the expected pattern of accents in the division type 2^3. This deviation creates a rhythmic tension known as *syncopation*.

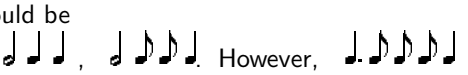
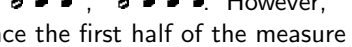
Syncopated rhythm is common in African-American and Latin American music. It is also highly developed in early music.

Examples. (1) Ghanaian pop song[3] (2) Telemann Oboe Sonata[4]

If the division type is binary, how many syncopated rhythm patterns of k notes are there?

Uniform Rhythm Patterns

Uniform rhythm patterns are definitely **not** syncopated. Let a *uniform* rhythm be any rhythm that results from a sequence of equal divisions of notes. For example,  is a uniform rhythm of division type 2^3. The corresponding sequence would be

 However,  is not uniform, since the first half of the measure has been divided in the ratio $3:1$.

Examples. (1) Telemann again (2) MIDI drums

The Catalan numbers got rhythm!

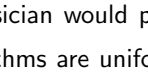
Theorem 1 (with P. Klingsberg) *The number of uniform rhythm patterns of binary division type with k notes is counted by the Catalan number $C(k-1) = (2k-2)!/(k!(k-1)!)$.*

Catalan numbers: $1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

Proof. See tree above!

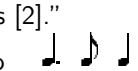
The same reasoning gives a formula for the number of uniform rhythm patterns of division type 2^d, where d is fixed. Other results on binary trees have interesting musical interpretations.

Are "Non-uniform" and "Syncopated" Equivalent?

All uniform rhythms are non-syncopated. However, a musician would probably not agree that all non-syncopated rhythms are uniform. For example,  does not sound syncopated.

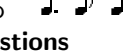
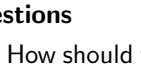
Example. Corelli Sonata[5]

According to the *New Harvard Dictionary of Music*, syncopation is "...a momentary contradiction of the prevailing meter or pulse. This may take the form of a temporary transformation of the fundamental character of the meter ...or it may be simply the contradition of the regular succession of strong and weak beats within a measure or group of measures [2]."

So  is not syncopated because long notes correspond to strong beats in division type 2^3.

Regular Rhythm

A rhythm is *regular* if the duration of notes starting on accented beats is as least as long as the duration of notes on weaker beats. All uniform rhythms are regular. Not all regular rhythms are uniform.

So  is regular;  is not!

Questions

- How should we define regular rhythm mathematically, so that "regular" is equivalent to "non-syncopated?"
- How can we count regular rhythm patterns?
- Can we define a "syncopation metric" expressing the level of syncopation in a rhythm pattern?
- What are the musical implications?

References

[1] Amulya Kumar Bag. Binomial theorem in ancient India. *Indian J. History Sci.*, 1:68–74, 1966.

[2] Don Randel, editor. *The New Harvard Dictionary of Music*. Belknap Press, Cambridge, MA, 1986.

[3] Parmanand Singh. Ācāraya Hemacandra and the (so-called) Fibonacci numbers. *Math. Ed. (Siwan)*, 20(1):28–30, 1986.

[3] E.T. Mensah, 205, *Rough Guide to West African Music*. The Rough Guides, RGNET 1002, 1995.

[4] Phillip Goodwin et al., Allegro, Trio XII in E♭ Major, *Oboe Sonatas*, Georg Philipp Telemann. Harmonia Mundi, HMU 907152, 1996.

[5] Chiara Banchini et al., Allegro, Sonata in A major, Arcangelo Corelli, *Sonate a Violino e Violone o Cimbalo op. V, Parte Prima*. Harmonia Mundi, HMC 901307, 1989.