The Math Behind the Music, by Leon Harkleroad, Cambridge University Press, 2006, 158 pages plus audio CD, £40.00 (70.00 USD) hardback (ISBN 0521810957), £14.99 (24.99 USD) paperback (ISBN 0521009359). MAA Outlooks series.

Perhaps mathematics and music are not as closely related as we think. The cliché 'math and music go together' is memorable for its seeming contradiction: how could a field that appears (to many people) cold, heartless, and abstract apply to something as enjoyable, emotional, and accessible as music? Recently, cognitive psychologist Daniel Levitin argued that musical talent is not inherently linked to mathematical ability [1]. He studied people with Williams syndrome who are often proficient musicians, yet incapable of doing mathematics (they typically have the intellectual ability of a young child).

Levitin's research is intriguing because the idea that mathematics and music are intimately connected has been widely accepted for centuries. Music, along with astronomy, was one of the first subjects to which mathematics was applied. The Pythagoreans' discovery that vibrating strings whose lengths form simple ratios produce pleasing harmonies was—and is—a compelling demonstration of the ubiquity of mathematics. The core mathematics curriculum in medieval universities included music. Several prominent mathematicians (Euler, for one) investigated mathematical phenomena in music. Musical notation—whether the modern staff system or the circular drumming notation from medieval Persia—is inherently geometrical. This fascination extended beyond the Western world; ancient Indian scholars also connected music and mathematics.

Nevertheless, the fact that we find some mathematical structure in music is not remarkable. It is said that mathematics is the science of pattern; patterns abound in the natural and man-made worlds; therefore, we expect to find mathematics in music. Statistics, mathematical biology, and dynamical systems are far more significant than the lowly mathematics of music. Why does the connection between music and math continue to fascinate us?

It is surprisingly difficult to answer this question. Mathematicians appear more excited than musicians by the thought that math and music are related. They seize the opportunity to demonstrate that math is 'beautiful' by drawing parallels between beauty in math and beauty in music. This ignores the obvious: although math has inspired some composers, music determined by rigid mathematical rules is not guaranteed to be 'beautiful', and rarely is. Moreover, music that purports to be algorithmically generated is, in fact, primarily determined by the composer's choices (math has its subjective side, but to a lesser extent). Perhaps in reaction to the subjectivity of music criticism, music theorists seem to relish the rigor and formalism—even pedantry—of mathematics (this tendency does not produce the best mathematical writing).

Even the list of central topics constituting 'mathematics and music' is not agreed upon. Mathematical music theory written by and for music theorists and composers differs substantially from material on music written by and for mathematicians. For example, few mathematicians—even those with a interest in music—have heard of neo-Riemannian set theory, an application of group theory to chords that has been a staple of music theory for the past few decades ('Riemann' refers to the music theorist Hugo Riemann, not the mathematician Bernhard Riemann). Likewise, music theorists may be unfamiliar with Fourier series, which partially explain the origin of scales and chords. An author who has training in both fields beyond an undergraduate level is unusual. Collaboration across the disciplines seems the best solution, but it is rare.

Despite all this, I, for one, remain convinced that there is a deep connection between math and music—a connection that goes beyond the presence of mathematical patterns in music. Although composers differ in the extent to which they explicitly use mathematics, composers and mathematicians both give form to abstract ideas. More than any other art form, the physical realization of music (i.e. sheet music) is not considered the actual music itself; in the same way, a mathematical object or idea exists independently of the way it is notated.

Several books written for mathematicians on the connections between math and music have appeared recently. Leon Harkleroad's book *The Math Behind the Music* is one of *two* such books published by Cambridge University Press in 2006. *The Math Behind the Music* purports to be 'a reputable source that [gathers] several of the important topics in one place and [gives] them a broadly accessible presentation'. Instead of going into depth, Harkleroad offers a tour of some of the applications of math to music most familiar to mathematicians, plus a few idiosyncratic topics such as change-ringing and contradancing. His book assumes only a light background in math and music, though he moves through topics like group theory fairly quickly for the general reader.

This book is a good introduction to some of the chief applications of math to music (at least, what mathematicians consider the chief applications). At 127 pages, plus endnotes and an accompanying CD, it's an easy read. Harkleroad doesn't assume too much of the reader—he introduces mathematical topics such as sine curves, groups, and probability. Basic music theory concepts such as intervals and chords are well explained, though he does assume the reader can read music. The CD provides audio for most of the examples in the book. *The Math Behind the Music* is suitable for use in an undergraduate liberal-arts math class. However, the treatment is too thin for an upper-level mathematics class (for that, I recommend Benson [2]).

The book opens with a discussion of the fundamentals of tone production: frequency, pitch, timbre, and overtones. The author makes an interesting point about our ears' ability to separate pure-tone components of a musical sound, which may explain why mathematicians naturally decompose musical sounds into sums of sinusoids. The second chapter, 'Tuning up', follows up nicely with a discussion of Pythagorean tuning, just intonations, and equal temperament.

Harkleroad's fourth chapter, 'How to vary a theme mathematically', is particularly strong. He introduces three symmetry operations: transposition (raising a melody by some number of pitches), inversion (exchanging movement upward in pitch with an equally-sized downward movement, and vice versa), and retrograde (playing a melody backwards). If we stipulate that transposition upwards by an octave is equivalent to the identity, since our ears naturally identify pitches that are an octave apart, each of these operations generates a cyclic group; the three operations together generate a nonabelian group of forty-eight elements containing several musically significant subgroups. These operations were used in twelve-tone compositional technique to generate tone rows. Harkleroad explains groups, subgroups, and cosets well. This material, with some supplementation by the instructor, would make an excellent introduction to groups for an undergraduate class.

The following chapter, 'Bells and groups', further develops connections between musical structure and groups with an explanation of the old English practice of change ringing, in which several players ring a sequence of patterns ('changes') on church bells. As the bells are extremely heavy and difficult to control, the possible sequences of patterns are restricted. For example, if we start by ringing six bells in order, represented by the pattern 123456, the next 'change' cannot require any bell to move more than one position forward or backward. Therefore, 214365 can follow 123456, but 321456 cannot. An entire sequence of changes may take several hours to perform. It starts and ends with the bells played in order; no other pattern may be repeated. Composing change-ringing sequences (many of which have been played for centuries) involves a surprising amount of group theory.

The use of probability theory in music makes several appearances: dice games popular in the late 1700s (basically, composing by numbers), and chance music by twentieth-century composers such as Hillier, Cage, and Xenakis. This is a topic that deserves to be better known to mathematicians.

The last three chapters are somewhat disappointing. Harkleroad covers a lot of ground (we hear about a simplistic experiment examining frequency of chord changes in popular songs; Voss and Clarke's famous paper on music and $1/f$ noise; the use of L-systems in composition; and the mathematics of contradancing), but his presentation of the topics could be better organized. The final chapter 'How NOT to mix music with mathematics' goes after two easy targets: Birkhoff's theory of aesthetic measure, and Lendvai's theory about Fibonacci numbers in Bartok. Although one must agree with Harkleroad about these two examples, I was hoping for a more provocative treatment of the subject.

Harkleroad's informality is sometimes refreshing; however, at other times his inattention to detail is frustrating. For example, in the last chapter, Harkleroad (justifiably) savages Lendvai's theory, but he does not include a bibliographical reference to Lendvai's work (for the record, it's [3]). Although Harkleroad's intention is not to write a scholarly book on the subject, it is still troubling that he neglects a lot of recent scholarship (for example, Sethares' groundbreaking research connecting the spectrum of musical sound and the timbre of various musical instruments with the pitches in scale [4]; Benson's excellent and exhaustive book, which has been available online for years [2]; and pretty much everything in recent mathematical music theory). There are other places where the author could have taken more care: for example, he introduces sine curves (p. 16) with 'In trigonometry, the scene of most people's first contact with sines, they show up as ratios of lengths of sides of right triangles. A nonmathematician might well wonder why sines would appear in the present context', yet he never explains this surprising connection. In addition, there are a number of typos, especially in figure captions.

Nevertheless, it is difficult to find a short book that pulls together a number of the chief applications of math to music that have been explored by mathematicians. *The Math Behind the Music* does this, and as such would be helpful to anyone who seeks an introduction to the field.

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References

[1] Levitin, D., 2006, *This is Your Brain on Music: The Science of a Human Obsession* (New York: Dutton Adult).

[2] Benson, D., 2006, *Music: A Mathematical Offering* (Cambridge: Cambridge University Press).

[3] Lendvai, E., 1971, *Béla Bartók: An Analysis of His Music* (London: Kahn and Averill).

[4] Sethares, W., 2004, *Tuning, Timbre, Spectrum, Scale*, 2nd edition (Berlin: Springer).