# THE $G$-SEQUENCE IN RATIONAL HOMOTOPY THEORY

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Abstract. We give an account of recent work of Gregory Lupton and the author on function spaces, the Gottlieb group, the evaluation subgroup and the $G$-sequence of a map within the framework of rational homotopy theory.

## 1. Introduction

The *Gottlieb group* $G_n(X)$ of a space $X$ is the subgroup of $\pi_n(X)$ consisting of classes $\alpha\colon S^n \to X$ such that the wedge $(\alpha \,|\, 1_X)\colon S^n \vee X \to X$ admits an extension $F\colon S^n \times X \to X$ to the product. Alternately, for reasonable, pointed spaces $X$,

$$G_n(X) = \omega_\sharp(\pi_n(\mathrm{map}(X, X; 1)))$$

where $\mathrm{map}(X, X; 1)$ is the path component of the identity map in the space of basepoint-free self-maps of $X$ and $\omega\colon \mathrm{map}(X, X; 1) \to X$ is the evaluation fibration $\omega(f) = f(*)$.

The study of the properties and structure of the Gottlieb groups represents a fundamental problem in homotopy theory dating back to their introduction by D. Gottlieb in the 1960's [6, 9]. Connections between the Gottlieb groups and fixed point theory [12, 6, 29] transformation groups [8, 33, 22] covering spaces [8, 13] and the homotopy theory of fibrations [7, 10, 27] have been extensively researched. While concrete calculations of the Gottlieb groups are difficult, examples do include the class of $H$-spaces [9], aspherical spaces [6], Stiefel manifolds [13], lens spaces [31] and homogeneous spaces [15]. Recently, M. Golasinski and J. Mukai have made complete calculations of $G_{n+k}(S^n)$ for a range of $k$ [5] by taking advantage of the identity $G_*(S^n) = P_*(S^n)$ where $P_*(X)$ denotes the Whitehead center of $X$. In [2], Y. Félix and S. Halperin obtained global results on the Gottlieb groups of simply connected rational spaces of finite L.S. category showing $G_{2n}(X_{\mathbb{Q}}) = 0$ and $\dim(G_{odd}(X_{\mathbb{Q}})) \leq \mathrm{cat}(X_{\mathbb{Q}})$. Since the Gottieb groups commute with localization for finite complexes [14], the Félix-Halperin results put strong restrictions on the infinite order component of the Gottlieb groups of finite simply connected complexes. These results notwithstanding, fundamental questions concerning the structure of the Gottlieb groups remain even after rationalization. For example, the following conjecture represents an open question.

**Conjecture 1.1.** ([3, p.518]) *Let $X$ be a finite complex of dimension $q$. Then the rational Gottlieb groups $G_n(X) \otimes \mathbb{Q} = 0$ for all $n \geq 2q - 1$.*

Conjecture 1.1, in turn, highlights a fundamental difficulty regarding the Gottlieb groups: their lack of functorality. A map of space $f\colon X \to Y$ does not generally induce a map on Gottlieb groups. Because of this fact, the usual inductive approaches

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to the study of homotopy invariants, via cellular or Postnikov decompositions, do not apply directly to the Gottlieb group. To obtain a functorial setting, it is natural to introduce the *evaluation subgroups* $G_n(Y, X; f)$ of a map $f\colon X \to Y$. Here $G_n(Y, X; f)$ consists of classes $\alpha\colon S^n \to Y$ such that the wedge $(\alpha \mid f)\colon S^n \vee X \to Y$ admits an extension $F\colon S^n \times X \to Y$ to the product. Alternately,

$$G_n(Y, X; f) = \omega_\sharp(\pi_n(\mathrm{map}(X, Y; f)))$$

where we again write $\omega\colon \mathrm{map}(X, Y; f) \to X$ for the evaluation fibration. It is easy to see that $f\colon X \to Y$ induces $f_\sharp\colon G_n(X) \to G_n(Y, X; f)$.

In a series of papers [16, 17, 18], K.Y. Lee and M.H. Woo showed the homomorphism $f_\sharp\colon G_*(X) \to G_*(Y, X; f)$ fits into an interesting boundary sequence called the *G-sequence* of the map $f\colon X \to Y$. The $G$-sequence of $f\colon X \to Y$ is a sequence of subgroups and restricted homomorphisms from the long exact homotopy sequence of the map $f$. It takes the form

$$\cdots \longrightarrow G_n(X) \xrightarrow{f_\sharp} G_n(Y, X; f) \longrightarrow G_n^{rel}(Y, X; f) \longrightarrow G_{n-1}(X) \longrightarrow \cdots$$

Since the homomorphisms in the $G$-sequence are restrictions of those in the long exact homotopy sequence of the map $f$, consecutive compositions in the $G$-sequence are trivial. The $G$-sequence is exact in some cases [19, 31] but nonexact in general [19]. The $G$-sequence, its various generalizations and their properties is an active area of current research [19, 20, 30, 31].

In Section 3 below, we describe the two frameworks of chain complexes constructed, in [23, 24], in terms of (generalized) derivations of Sullivan and Quillen minimal models, respectively. The main results of these papers show the homotopy theory of function spaces is modeled, at the level of raional homotopy groups, by the homology theory of these chain complexes. In particular, the rationalization of the $G$-sequence derives from standard homology exact sequences within these frameworks. We describe these results in Section 4. We then discuss progress made on the following basic problems regarding rationalized evaluation subgroups and the rationalized $G$-sequence:

**Problem 1.2.** *Analyze the exactness of the rationalized $G$-sequence of a cellular inclusion.*

The question of exactness of the $G$-sequence of a cellular inclusion represents the first area of research concerning the $G$-sequence (see [16, 17]). In [24, §5], we give a complete analysis of the exactness of the rationalized $G$-sequence at the Gottlieb term in the first nontrivial case. We recall this result in Section 5 below.

**Problem 1.3.** *Analyze the exactness of the rationalized $G$-sequence of a fibre inclusion as a measure of the relative fibre homotopy triviality of the fibration.*

Gottlieb established an important connection between the Gottlieb group and the theory of fibrations in [7] where he identified the Gottlieb groups as the image of the linking homomomosphism in the long exact homotopy sequence of the universal fibration $X \to UX \to \mathrm{Baut}(X)$ with fibre $X$. As a consequence, he obtained the identity

$$G_*(X) = \bigcup_{\xi} \mathrm{im}\{\partial_\xi\}$$

where the union is taken over fibrations $\xi\colon X \to E \to B$ with linking homomorphism $\partial_\xi\colon \pi_{*+1}(B) \to \pi_*(X)$. Thus the vansihing of the Gottlieb group implies the weak homotopy triviality of all $X$-fibrations. In [25], we consider the exactness of the $G$-sequence of a fibre inclusion $j\colon X \to E$ as a measure of the relative triviality of a fibration $\xi\colon X \xrightarrow{j} E \xrightarrow{p} B$. There we give a characterization of this notion for rational fibrations in terms of the map induced on rational homotopy groups by the classifying map $h\colon B \to \mathrm{Baut}(X)$. In particular, the exactness of the $G$-sequence of a fibre inclusion gives an equivalent phrasing of a famous conjecture of Halperin in rational homotopy theory [23, Th.4.9]. We describe these results in Section 6 below.

**Problem 1.4.** *Describe the difference between the evaluation subgroup $G_*(Y, X; f)$ and the Whitehead centralizer $P_*(Y, X; f)$ of a map $f\colon X \to Y$ where*

$$P_n(Y, X; f) = \{\alpha \in \pi_n(Y) | [\alpha, f_\sharp(\beta)] = 0 \ \ \forall \beta \in \pi_*(X)\}.$$

The inclusion $G_*(Y, X; f) \subseteq P_*(Y, X; f)$ is direct from definitions. In [6], Gottlieb asked for a space $X$ for which $G_1(X) \neq P_1(X)$. The first such example was given by Ganea [4]. Oprea gave an example of inequality with a finite complex in [28]. The generalization of Gottlieb's question asks for a map $f\colon X \to Y$ such that $G_*(Y, X; f) \neq P_*(Y, X; f)$. In [24, §5], we identify the quotient group $G_n(Y, X; f)/P_n(Y, X; f)$ for rational spaces in terms of Quillen models and generalized adjoint maps. We describe this result in Section 7. Finally, in Section 8 we describe some consequences of our work for the rational homotopy theory of function spaces.

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## 2. The $G$-sequence of a map.

Following [18], the $G$-sequence of a map $f\colon X \to Y$ between CW complexes may be constructed as follows: Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(X, X; 1) & \xrightarrow{f_*} & \mathrm{map}(X, Y; f) \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $f_*$ denotes composition with $f$. The induced homomorphisms of homotopy groups $(f_*)_\#$ and $f_\#$ form part of the long exact homotopy sequences of the maps $f_*$ and $f$ respectively. The evaluation maps induce maps of each term in these long exact sequences, resulting in a homotopy ladder. The $G$-sequence of the map $f$ is then the image of the top long exact homotopy sequence in that of the bottom:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \pi_{n+1}(f_*) & \longrightarrow & \pi_n(\mathrm{map}(X,X;1)) & \xrightarrow{(f_*)_\#} & \pi_n(\mathrm{map}(X,Y;f)) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \omega_\#} & & \downarrow{\scriptstyle \omega_\#} & & \downarrow{\scriptstyle \omega_\#} & \\
\cdots \longrightarrow & G^{rel}_{n+1}(Y,X;f) & \longrightarrow & G_n(X) & \xrightarrow{f_\#} & G_n(Y,X;f) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_{n+1}(f) & \longrightarrow & \pi_n(X) & \xrightarrow{f_\#} & \pi_n(Y) & \longrightarrow \cdots
\end{array}$$

The construction introduces a new group $G^{rel}_{n+1}(Y,X;f)$, the "relative" term, which is the image in $\pi_{n+1}(f)$ of the homomorphism induced by the evaluation maps. Since the maps in the $G$-sequence are restrictions of the maps in the long exact homotopy sequence of the map $f$, the $G$-sequence is half-exact, i.e., successive compositions vanish.

We may also construct the $G$-sequence as a "kernel" sequence by taking a step back in the Barratt-Puppe sequence of the evaluation fibrations $\omega\colon \mathrm{map}(X,X;1) \to X$ and $\omega\colon \mathrm{map}(X,Y;f) \to Y$. Let $\mathrm{map}_*(X,Y;f)$ denote the component of $f$ in the space of basepoint preserving maps from $X$ to $Y$. Recall a fibration $X \stackrel{j}{\to} E \stackrel{p}{\to} B$ gives rise to a fibre sequence $\Omega B \stackrel{\partial}{\to} X \stackrel{j}{\to} E$ where the map $\partial$ induces the linking homomorphism in the long exact homotopy sequence via the identification $\pi_*(\Omega B) \cong \pi_{*+1}(B)$. Thus given a (based) map $f\colon X \to Y$ we have a commutative square

$$\begin{array}{ccc}
\Omega X & \xrightarrow{\Omega f} & \Omega Y \\
{\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\
\mathrm{map}_*(X,X;1) & \xrightarrow{f_*} & \mathrm{map}_*(X,Y;f),
\end{array}$$

in which the vertical maps are the transgression maps from the respective evaluation fibrations. The $G$-sequence of the map $f$ may be defined, with a shift in degree, as the kernel sequence of the above homotopy ladder as shown here here:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & G_{n+1}(X) & \xrightarrow{(\Omega f)_\#} & G_{n+1}(Y,X;f) & \longrightarrow & G^{rel}_{n+1}(Y,X;f) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_n(\Omega X) & \xrightarrow{(\Omega f)_\#} & \pi_n(\Omega Y) & \longrightarrow & \pi_n(\Omega f) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \partial_\#} & & \downarrow{\scriptstyle \partial_\#} & & \downarrow{\scriptstyle \Delta} & \\
\cdots \longrightarrow & \pi_n(\mathrm{map}_*(X,X;1)) & \xrightarrow{(f_*)_\#} & \pi_n(\mathrm{map}_*(X,Y;f)) & \longrightarrow & \pi_n(f_*) & \longrightarrow \cdots
\end{array}$$

In Section 4 we identify the rational homotopy data in the "image sequence" construction of the $G$-sequence within the framework of Sullivan minimal models and the data in the "kernel sequence" construction within the framework of Quillen minimal models.

## 3. Generalized Derivations of DG Algebras and DG Lie Algebras

In this section, we describe the two frameworks of chain complexes of generalized derivations of DG (Lie) algebras. We define Gottlieb groups, evaluation subgroups and the $G$-sequence within these frameworks.

First fix a DG algebra map $\phi\colon (A, d_A) \to (B, d_B)$. Here we will assume our DG algebras are simply connected and, very often, *minimal*, i.e. free with decomoposable differential. A *$\phi$-derivation* of degree $n$ is a linear map $\theta\colon A \to B$ reducing degree by $n$ and satisfying the identity

$$\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y).$$

Let $\mathrm{Der}_n(A, B; \phi)$ denote the space of all $\phi$-derivations of degree $n$. When $n = 1$ we restrict to derivations $\theta$ with $d_B \circ \theta = -\theta \circ d_A$. Define a differential by the rule

$$D(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A.$$

In the special case $A = B$ and $f = 1_B$ we recover the ususal DG Lie algebra of derivations $\mathrm{Der}_n(B) = \mathrm{Der}_n(B, B; 1)$. Observe that the map $\phi\colon A \to B$ induces a chain map: $\phi^*\colon \mathrm{Der}_n(B) \to \mathrm{Der}_n(A, B; \phi)$ obtained by pre-composition with $\phi$.

To construct the evaluation subgroup of $\phi$ in this context, we consider the augmentation $\varepsilon\colon B \to \mathbb{Q}$ of the DG algebra $B$. Viewing $\mathbb{Q}$ as the trivial DG algebra concentrated in degree zero, composition with $\varepsilon$ induces a chain map $\varepsilon_*\colon \mathrm{Der}_*(A, B; \phi) \to \mathrm{Der}(A, \mathbb{Q}; \varepsilon)$ We define the *evaluation subgroup of* $\phi$ by setting

$$G_n(A, B; \phi) = \mathrm{im}\{H(\varepsilon_*)\colon H_n(\mathrm{Der}(A, B; \phi)) \to H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon))\}$$

The *Gottlieb group* of the DG algebra $(B, d_B)$ is obtained as a special case:

$$G_n(B) = \mathrm{im}\{H(\varepsilon_*)\colon H_n(\mathrm{Der}(B)) \to H_n(\mathrm{Der}(B, \mathbb{Q}; \varepsilon))\}$$

To construct the $G$-sequence of the DG algebra map $\phi\colon A \to B$, we recall the mapping cone construction for a chain map $\psi : (V, d_V) \to (W, d_W)$ of DG spaces. This is the DG space $(C_*(\psi), \delta)$ with $C_n(\psi) = V_{n-1} \oplus W_n$ and differential $\delta(v, w) = (-d_V(v), \psi(v) + d_W(w))$. We apply this construction to the chain maps $\phi^*\colon \mathrm{Der}_*(B) \to \mathrm{Der}_*(A, B; \phi)$ and $\widehat{\phi^*}\colon \mathrm{Der}_*(B, \mathbb{Q}) \to \mathrm{Der}(A, \mathbb{Q})$ both obtained by pre-composition with $\phi$. We define the *$G$-sequence of the DG algebra map* $\phi\colon A \to B$ to be the image sequence in the resulting commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{n+1}(C(\phi^*)) & \longrightarrow & H_n\big(\mathrm{Der}(B)\big) & \xrightarrow{H(\phi^*)} & H_n\big(\mathrm{Der}(A, B; \phi)\big) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle H(\varepsilon_*, \varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & \\
\cdots \longrightarrow & G^{rel}_{n+1}(A, B; \phi) & \longrightarrow & G_n(B) & \longrightarrow & G_n(A, B; \phi) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H_{n+1}(C(\widehat{\phi^*})) & \longrightarrow & H_n\big(\mathrm{Der}(B, \mathbb{Q}; \varepsilon)\big) & \xrightarrow{H(\widehat{\phi^*})} & H_n\big(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)\big) & \longrightarrow \cdots
\end{array}$$

Next fix $\phi\colon (K, d_K) \to (L, d_L)$ a map of DG Lie algebras. Here we assume a DG Lie algebra is at least connected and often minimal. Define a *$\phi$-derivation* of degree $n$ to be a linear map $\theta\colon K \to L$ increasing degree by $n$ and satisfying the identity

$$\theta([x, y]) = [\theta(x), \phi(y)] + (-1)^{n|x|}[\phi(x), \theta(y)].$$

Write $\mathrm{Der}_n(K, L; \phi)$ for the vector space of degree $n$ $\phi$-derivations (with the same restrictions in degree 1) and define the differential as before: $D(\theta) = d_L \circ \theta -$

$(-1)^{|\theta|}\theta \circ d_K$. We write $\mathrm{Der}_n(L) = \mathrm{Der}_n(L, L; 1)$ for the usual DG Lie algebra of derivations.

We introduce the *generalized adjoint map* corresponding to $\phi\colon L \to K$. This is a chain map $\mathrm{ad}_\phi\colon L_n \to \mathrm{Der}_n(K, L; \phi)$ given by $\mathrm{ad}_\phi(y)(x) = [\phi(x), y]$. Note that when $\phi = 1_L$ we recover the usual adjoint map $\mathrm{ad} : L \to \mathrm{Der}(L)$. We define the *evaluation subgroup of* $\phi$ by

$$G_{n+1}(K, L; \phi) = \ker\{H(\mathrm{ad}_\phi)\colon H_n(L) \to H_n(\mathrm{Der}(K, L; \phi))\}$$

and, as usual, the special case of the *Gottlieb group*

$$G_{n+1}(L) = \ker\{H(\mathrm{ad})\colon H_n(L) \to H_n(\mathrm{Der}(L))\}.$$

The *G-sequence of the DG Lie algebra map* $\phi\colon K \to L$ is now defined as the kernel sequence indicated in the diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & G^{rel}_{n+1}(L, K; \phi) & \longrightarrow & G_n(L) & \longrightarrow & G_n(L, K; \phi) & \longrightarrow \cdots \\
& \big\downarrow & & \big\downarrow & & \big\downarrow & \\
\cdots \longrightarrow & H_n(C(\phi)) & \longrightarrow & H_{n-1}(L) & \xrightarrow{H(\phi)} & H_{n-1}(K) & \longrightarrow \cdots \\
& \big\downarrow {\scriptstyle H(\mathrm{ad},\mathrm{ad}_\phi)} & & \big\downarrow {\scriptstyle H(\mathrm{ad})} & & \big\downarrow {\scriptstyle H(\mathrm{ad}_\phi)} & \\
\cdots \longrightarrow & H_n(C(\phi_*)) & \longrightarrow & H_{n-1}\big(\mathrm{Der}(L)\big) & \xrightarrow{H(\phi_*)} & H_{n-1}\big(\mathrm{Der}(L, K; \phi)\big) & \longrightarrow \cdots
\end{array}$$

## 4. The Rationalized $G$-Sequence

We state the main results from [23] and [24] which identify the rational homotopy data involved in defining the $G$-sequence within the frameworks of chain complexes of generalized derivations defined above. Given a map $f\colon X \to Y$ we write $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$ for the Sullivan minimal model of $f$ and $\mathcal{L}_f\colon \mathcal{L}_X \to \mathcal{L}_Y$ for the Quillen minimal model for $f$.

**Theorem 4.1.** [23, Th.2.1] *Let $X$ and $Y$ be simply connected CW complexes of finite type, with $X$ finite. For $n \geq 2$,the commutative squares*

$$\begin{array}{ccc}
\pi_n\big(\mathrm{map}(X, X; 1)\big) \otimes \mathbb{Q} & \xrightarrow{(f_*)_\# \otimes \mathbb{Q}} & \pi_n\big(\mathrm{map}(X, Y; f)\big) \otimes \mathbb{Q} \\
{\scriptstyle \omega_\# \otimes \mathbb{Q}} \big\downarrow & & \big\downarrow {\scriptstyle \omega_\# \otimes \mathbb{Q}} \\
\pi_n(X) \otimes \mathbb{Q} & \xrightarrow[f_\# \otimes \mathbb{Q}]{} & \pi_n(Y) \otimes \mathbb{Q}
\end{array}$$

*and*

$$\begin{array}{ccc}
H_n\big(\mathrm{Der}(\mathcal{M}_X)\big) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n\big(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)\big) \\
{\scriptstyle H(\varepsilon_*)} \big\downarrow & & \big\downarrow {\scriptstyle H(\varepsilon_*)} \\
H_n\big(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)\big) & \xrightarrow[H(\widehat{(\mathcal{M}_f)^*})]{} & H_n\big(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)\big)
\end{array}$$

*are equivalent for each $n \geq 2$.*

A consequence of this result is

**Theorem 4.2.** [23, Th.3.8] *The $G$-sequence of a map $f\colon X \to Y$ of simply connected finite type CW complexes with $X$ finite is equivalent to the $G$-sequence of the Sullivan minimal model $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$ of $f$.*

On the Quillen model side we have

**Theorem 4.3.** [24, Th.4.1] *Let $X$ and $Y$ be simply connected CW complexes of finite type, with $X$ finite. For $n \geq 2$,the commutative squares*

$$\begin{array}{ccc} \pi_n(\Omega X)\otimes\mathbb{Q} & \xrightarrow{(\Omega f)_\#\otimes\mathbb{Q}} & \pi_n(\Omega Y)\otimes\mathbb{Q} \\ \downarrow{\scriptstyle \partial\otimes\mathbb{Q}} & & \downarrow{\scriptstyle \partial\otimes\mathbb{Q}} \\ \pi_n\big(\mathrm{map}_*(X,X;1)\big)\otimes\mathbb{Q} & \xrightarrow[(f_*)_\#\otimes\mathbb{Q}]{} & \pi_n\big(\mathrm{map}_*(X,Y;f)\big)\otimes\mathbb{Q} \end{array}$$

*and*

$$\begin{array}{ccc} H_n(\mathcal{L}_X) & \xrightarrow{H(\mathcal{L}_f)} & H_n(\mathcal{L}_Y) \\ {\scriptstyle H(\mathrm{ad})}\downarrow & & \downarrow{\scriptstyle H(\mathrm{ad}_{\mathcal{L}_f})} \\ H_n\big(\mathrm{Der}(\mathcal{L}_X)\big) & \xrightarrow[H(\mathcal{L}_f)^*)]{} & H_n\big(\mathrm{Der}(\mathcal{L}_X,\mathcal{L}_Y;\mathcal{L}_f)\big) \end{array}$$

*are equivalent for each $n \geq 2$.*

and its consequence

**Theorem 4.4.** [24, Cor.6.4] *The $G$-sequence of a map $f\colon X \to Y$ of simply connected finite type CW complexes with $X$ finite is equivalent to the $G$-sequence of the Quillen minimal model $\mathcal{L}_f\colon \mathcal{L}_X \to \mathcal{L}_Y$ of $f$.*

Theorem 4.1 recovers, in particular, the characterization of the rationalized Gottlieb groups due to Félix and Halperin [2] as corresponding to the indecomposable elements in the Sullivan model whose dual homomorphisms extend to derivation cycles. Similarly, Theorem 4.3 recovers the identification of the rationalized Gottlieb groups due to D. Tanré [32] as the kernel of the map induced on homology by the Quillen model adjoint map.

## 5. (Non)exactness of the $G$-sequence

Using the results above, it is possible to effectively analyze the exactness of the $G$-sequence in many situations. We mention two results obtained in [23, 24], the first proved using Sullivan models and the second using Quillen models.

**Theorem 5.1.** [23, Ex.4.1] *There exists a map $f\colon X \to Y$ between finite simply connected complexes such that the rationalized $G$-sequence of $f$ is nonexact in a particular degree at all three types of terms occuring in the $G$-sequence.*

In fact, the map is quite simple. We remark that inexactness of the rationalized $G$-sequence implies inexactness of the original $G$-sequence. To take this further, following Lee and Woo [18], define the *$\omega$-homology groups* of a map $f$ by

$$H_n^{a\omega}(Y,X;f) = \frac{\ker\{f_\sharp\colon G_n(X) \to G_n(Y,X;f)\}}{\mathrm{im}\{\delta\colon G_{n+1}^{rel}(Y,X;f) \to G_n(X)\}}$$

with $H_n^{b\omega}(Y,X;f)$ and $H_n^{c\omega}(Y,X;f)$ defined, similarly, to be the homology of the $G$-sequence at the $G_n(Y,X;f)$ and $G_n^{rel}(Y,X;f)$ terms, respectively.

**Corollary 5.2.** *There exists a map $f\colon X \to Y$ between finite simply connected complexes with $H_*^{a\omega}(Y, X; f), H_*^{b\omega}(Y, X; f)$ and $H_*^{c\omega}(Y, X; f)$ all nontrivial (in fact, infinite) in particular degrees.*

The next result analyzes the issue of exactness for a cellular attachment in the first nontrivial case.

**Theorem 5.3.** [24, Th.6.7] *Let $X$ be a finite CW complex, and $Y = X \cup_\alpha e^{m+1}$ for some $\alpha \in \pi_m(X)$. Then the rational $G$-sequence of the inclusion $i\colon X \to Y$ is nonexact at the Gottlieb term if and only if all of the following three conditions hold.*

(1) $\alpha_{\mathbb{Q}} \neq 0$;
(2) $\alpha_{\mathbb{Q}} \in G_m(X_{\mathbb{Q}})$;
(3) *$Y$ is not rationally equivalent to a point.*

## 6. The $G$-Sequence of a Fibre Inclusion

In this section, we discuss some results from [25]. We view the $G$-sequence of a fibre inclusion as a measure of the relative fibre homotopy trivality of a fibration. Say a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ is *Gottlieb trivial* if the $G$-sequence of the fibre inclusion $j$ breaks into short exact sequences:

$$0 \to G_n(X) \to G_n(E, X; j) \to \pi_n(B) \to 0.$$

A fibre homotopy trivial fibration is Gottlieb trivial by [19, 34]. In fact, for all fibrations

$$\text{fibre homotopy trivial} \implies \text{Gottlieb trivial} \implies \text{weak homotopy trivial}$$

with both implications strict [25].

Given a fibration $\xi\colon X \xrightarrow{j} E \xrightarrow{p} B$ of simply connected complexes we consider the classifying map $h\colon B \to \mathrm{Baut}_1(X)$ where $\mathrm{aut}_1(X) = \mathrm{map}(X, X; 1)$. Here $\mathrm{Baut}_1(X)$ is the Dold-Lashof classifying space of the monoid $\mathrm{aut}_1(X)$. The map $h$ induces

$$h_\sharp \otimes \mathbb{Q}\colon \pi_*(B) \otimes \mathbb{Q} \to \pi_*(\mathrm{Baut}_1(X)) \otimes \mathbb{Q} \cong H_{n-1}(\mathrm{Der}(\mathcal{M}_X)).$$

In [25] we describe this map within the Sullivan model framework described in Sections 3 and 4. Gottlieb triviality is then characterized, for certain rational fibrations, by the following result.

**Theorem 6.1.** [25] *Let $\xi\colon X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected finite type CW complexes with $X$ finite. Suppose the rational homotopy groups of $B$ are finite-dimensional. Then $\xi$ is rationally Gottlieb trivial if and only if the classifying map $h\colon B \to \mathrm{Baut}_1(X)$ vanishes on rational homotopy groups.*

As shown in [23, §4], consideration of the $G$-sequence of a fibre inclusion provides a surprising link between the $G$-sequence and a famous conjecture of Halperin in rational homotopy theory. We say a space $X$ is an $F_0$*-space* if $X$ is simply connected elliptic complex (i.e. having finite-dimensional rational homotopy and cohomology) with $H^{odd}(X, \mathbb{Q}) = 0$. While the structure of the Sullivan model of an $F_0$-space is quite simple, the following conjecture due to Halperin is open:

**Conjecture 6.2.** [3, p.516] *The rational Serre spectral sequence collapses for all fibrations with fibre an $F_0$-space.*

Results of W. Meier [26] and Lupton [21] imply Conjecture 6.2 holds if and only if every fibration of the form $X \to E \to S^{2n+1}$ with $X$ an $F_0$-space is rationally fibre homotopy trivial. Note that a fibration over a sphere is rationally fibre homotopy trivial if and only if the classifying map vanishes on rational homotopy groups. Thus we have

**Corollary 6.3.** [23, Cor.4.18] *Let $X$ be an $F_0$-space. Then Halperin's conjecture holds for $X$ if and only if every fibration $X \to E \to S^{2n+1}$ is rationally Gottlieb trivial.*

## 7. Gottlieb's Question in Rational Homotopy Theory

Using the Quillen model description of the rationalized evaluation subgroup, we address the generalization of Gottlieb's question concerning the difference between the evaluation subgroup and the Whitehead centralizer of map $f\colon X \to Y$. Consider the commutative diagram:

$$\begin{array}{ccc} H_n(\mathcal{L}_Y) & \xrightarrow{H(\mathrm{ad}_{\mathcal{L}_f})} & H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ & \searrow^{\mathrm{ad}_{H(\mathcal{L}_f)}} & \downarrow I \\ & & \mathrm{Der}_n(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f)). \end{array}$$

Here $I$ is the "induced derivation" map defined as follows: Given a derivation cycle $\theta \in \mathrm{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ and a cycle $\xi \in \mathcal{L}_X$ we set $I(\langle\theta\rangle)(\langle\xi\rangle) = \langle\theta(\xi)\rangle$. It is direct to check $I$ is well-defined. Since $P_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong \ker\{\mathrm{ad}_{H(\mathcal{L}_f)}\}$ and $G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = \ker\{H(\mathrm{ad}_{\mathcal{L}_f}\}$ we obtain

**Theorem 7.1.** [24, Th.5.4] *Let $f\colon X \to Y$ be a map between simply connected CW complexes of finite type.*

$$\frac{P_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})}{G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})} \cong \ker(I) \cap \mathrm{im}(H(\mathrm{ad}_{\mathcal{L}_f})).$$

Using this result, it is easy to construct examples of maps $f\colon X \to Y$ for which $G_*(Y_{\mathbb{Q}}, X_Q; f_{\mathbb{Q}}) \neq P_*(Y_{\mathbb{Q}}, X_Q; f_{\mathbb{Q}})$. In the other direction, we have

**Theorem 7.2.** [24, Th.5.9] *Let $f\colon X \to Y$ be a coformal map between simply connected coformal CW complexes of finite type. Then*

$$G_*(Y_{\mathbb{Q}}, X_Q; f_{\mathbb{Q}}) = P_*(Y_{\mathbb{Q}}, X_Q; f_{\mathbb{Q}}).$$

Properties of the induced derivation homomorphism $I$ and its analogue for Sullivan models are directly related to the rationalized $G$-sequence (see [23, Th.4.3]). The induced derivation map for a Sullivan model has also appeared in recent work in geometry [1].

## 8. Rational Homotopy Groups of Function Spaces

We conclude by observing some consequences of Theorems 4.1 and 4.3 for the homotopy theory of function spaces at the level of rational homotopy groups. These results imply identifications of the standard long exact homotopy homotopy sequences which arise with function spaces. In particular, we obtain two descriptions of the rationalization of the long exact sequence

$$\cdots \to \pi_{n+1}(Y) \to \pi_n(\mathrm{map}_*(X,Y;f)) \to \pi_n(\mathrm{map}(X,Y;f)) \to \cdots \tag{1}$$

of the evaluation fibration

$$\mathrm{map}_*(X,Y;f) \to \mathrm{map}(X,Y;f) \xrightarrow{\omega} Y.$$

In the framework of Sullivan models, the rationalization of (1) takes the form

$$\cdots \longrightarrow H_{n+1}\big(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)\big) \to H_n\big(\widetilde{\mathrm{Der}}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)\big) \to H_n\big(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)\big) \xrightarrow{H(\varepsilon_*)} \cdots$$

[23, Th.3.12]. Here

$$\widetilde{\mathrm{Der}}_n(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f = \ker\{\varepsilon_* \colon \mathrm{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \to \mathrm{Der}_*(\mathcal{M}_Y, \mathbb{Q}; \varepsilon).\}$$

On Quillen models, the rationalization of (1) is equivalent to the sequence

$$\cdots \longrightarrow H_n\big(\mathcal{L}_Y\big) \xrightarrow{H(\mathrm{ad}_{\mathcal{L}_f})} H_n\big(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)\big) \to H_n\big(C(\mathrm{ad}_{\mathcal{L}_f})\big) \longrightarrow \cdots$$

[24, Th.4.4]. We may also obtain identifications of the long exact rational homotopy sequences of the induced maps

$$g_* \colon \mathrm{map}(X,Y;f) \to \mathrm{map}(X,Y';g\circ f) \quad \text{and} \quad h^* \colon \mathrm{map}(X,Y;f) \to \mathrm{map}(X',Y;f\circ h)$$

corresponding to maps $g\colon Y \to Y'$ and $h\colon X' \to X$ in both settings by using the mapping cone as described in Section 3.

Finally, Theorems 4.1 and 4.3 have direct consequences for the rational homotopy of function spaces. For example, combined with an argument of P.-P. Grivel [11], Theorem 4.1 implies the even rational homotopy groups of $\mathrm{map}(X,Y;f)$ for a map $f\colon X \to Y$ between $F_0$-spaces depend only upon the map induced by $f$ on rational cohomology.

**Theorem 8.1.** [23, Cor.3.12] *Let $f\colon X \to Y$ be a map between $F_0$-spaces. Then*

$$\pi_{2k}(\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \cong H_{2k}\big(\mathrm{Der}(H^*(Y,\mathbb{Q}), H^*(X,\mathbb{Q}); H(f)\big).$$

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