CARDINALITY OF THE SET OF REAL FUNCTIONS WITH A GIVEN CONTINUITY SET

JIAMING CHEN* AND SAM SMITH†

Abstract. Expanding on an old result of W. H. Young, we determine the cardinality of the set of functions $f\colon X \to \mathbb{R}$ with continuity set belonging to a given collection of subsets of X, where $X = \mathbb{R}$ or, more generally, X a complete, separable metric space.

1. Introduction

Let X be a metric space and $f\colon X \to \mathbb{R}$ an arbitrary function. Let C_f denote the subset of X consisting of those points x such that f is continuous at x. It is a standard analysis problem to determine the set C_f for a given function f. A more interesting problem arises by turning the tables: Given a subset $A \subseteq X$ we ask for a function $f\colon X \to \mathbb{R}$ with continuity set $C_f = A$. In particular, we have the existence question:

Question. *For which subsets $A \subseteq X$ does there exist a function $f\colon X \to \mathbb{R}$ with $C_f = A$?*

As described in the text [1], consideration of this question for the case $X = \mathbb{R}$ follows an interesting historical path through elementary analysis. The story begins with Dirichlet's construction of a nowhere continuous function:

$$D(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

The construction of a function $f\colon \mathbb{R} \to \mathbb{R}$ with C_f an open interval $A = (a, b)$ (or, for that matter, A an open subset of $\mathbb{R}$) is a direct consequence:

$$f(x) = \begin{cases} 0 & x \in A \\ D(x) & x \notin A. \end{cases}$$

That this construction does not work for A a closed interval reveals the topological aspect of the question. A function $g\colon \mathbb{R} \to \mathbb{R}$ with $C_g = [a, b]$ can be obtained, however, by damping the Dirichlet function at the endpoints:

$$g(x) = \begin{cases} 0 & x \in [a, b] \\ (x-a)D(x) & x < a \\ (x-b)D(x) & x > b. \end{cases}$$

The construction of a function with continuity set the irrationals was given by Thomae (see [1, p.99]). On the other hand, the fact that there are no functions $f\colon \mathbb{R} \to \mathbb{R}$ with $C_f = \mathbb{Q}$ is a deeper result involving the first notions of Baire category (see Example 2.8 below and [1, p.111]).

*†Saint Joseph's University, Philadelphia, PA.

The question was completely resolved for $X = \mathbb{R}$ by W. H. Young at the turn of the last century (see [3, pgs. 313-316]). Define a subset $E \subseteq X$ to be a *G_δ-set* if there are countably many open subsets $O_1, O_2, \cdots$ such that

$$E = \bigcap_{n=1}^{\infty} O_n.$$

It is easy to see that, in $\mathbb{R}$, the G_δ sets include the open sets (trivially), closed intervals and any set whose complement is countable (e.g., the irrationals). Young proved that the continuity sets C_f for $f: \mathbb{R} \to \mathbb{R}$ correspond exactly to the class of G_δ-sets in $\mathbb{R}$. The key step in the proof is the construction of a function with continuity set an arbitrary G_δ-set. Recently S. S. Kim [4] generalized Young's characterization of continuity sets to the case $f: X \to \mathbb{R}$ for X a metric space.

Our purpose in this paper is to address the following related set-theoretic problem:

Problem. *Given a collection $\mathcal{A}$ of subsets of X determine the cardinality of the set of functions $f: X \to \mathbb{R}$ with C_f belonging to $\mathcal{A}$.*

We solve this problem in Section 3 for X a complete, separable metric space. We assume familiarity with the basic notions concerning the topology of metric spaces (e.g., open sets, continuity, limit points and closure). References for this material include [5, 6]. Our exposition is self-contained with one exception — we assume the Baire Category Theorem for complete metric spaces (Theorem 2.5, below). Finally, we assume familiarity with basic set theory (e.g., countablility, power sets, the Continuum Hypothesis). We prove the set-theoretic results we need in Section 2.

2. Cardinality of some sets of functions

We begin with some notation. Given sets A and B we write B^A for the set of functions from a set A to a set B and $\mathcal{P}(A)$ for the power set of A. Given a metric space X and a function $f: X \to \mathbb{R}$, the assignment $f \mapsto C_f$ determines a function

$$\mathcal{C}_X: \mathbb{R}^X \to \mathcal{P}(X)$$

given by $\mathcal{C}_X(f) = C_f$. We focus on the preimages of subsets of $\mathcal{A} \subseteq \mathcal{P}(X)$. By definition,

$$\mathcal{C}_X^{-1}(\mathcal{A}) = \{f: X \to \mathbb{R} \,|\, C_f = A \quad \text{for some} \quad A \in \mathcal{A}\}.$$

Given a set Z, we write $\text{card}(Z)$ for the cardinality of Z. In particular, we write $c = \text{card}(\mathbb{R})$ and $2^c = \text{card}(\mathcal{P}(\mathbb{R}))$. We assume, for the present, the

Continuum Hypothesis. *An uncountable subset $A \subseteq \mathbb{R}$ has cardinality c.*

We will need the following standard fact concerning the arithmetic of cardinals.

Theorem 2.1. *Let $A, B \subseteq \mathbb{R}$ be subsets with A infinite and B containing at least 2 points. Then*

$$\text{card}\big(B^A\big) = \begin{cases} c & \text{if } A \text{ is countable} \\ 2^c & \text{if } A \text{ is uncountable.} \end{cases}$$

Proof. A function $f: A \to P(B)$ gives rise to $D(f) = \{(a, b) | a \in A, b \in f(a)\}$ a subset of $A \times B$. The assignment $f \mapsto D(f)$ gives a injection $D: P(B)^A \to P(A \times B)$. Suppose A is countable. Since B has two points, we know B^A has cardinality at least c. We may thus assume the worst case scenario, $B = \mathbb{R} \cong \mathcal{P}(\mathbb{N})$ where $\mathbb{N} = \{1, 2, \ldots\}$. We then obtain

$$\text{card}\left(B^A\right) = \text{card}\left(\mathcal{P}(\mathbb{N})^A\right) \leq \text{card}\left(\mathcal{P}(A \times \mathbb{N})\right) = \text{card}\left(\mathcal{P}(\mathbb{N})\right) = c.$$

Next given $f\colon A \to B$ let $\Delta(f) = \{(a, f(a)) | a \in A\} \subseteq A \times B$ denote the diagonal of f. The assignment $f \mapsto \Delta(f)$ defines an injection $\Delta\colon B^A \to \mathcal{P}(A \times B)$. If A is uncountable, then

$$\text{card}\left(B^A\right) \leq \text{card}\left(\mathbb{R}^{\mathbb{R}}\right) \leq \text{card}\left(\mathcal{P}(\mathbb{R} \times \mathbb{R})\right) = \text{card}\left(\mathcal{P}(\mathbb{R})\right) = 2^c.$$

The reverse inequality is clear. □

We will also need the following consequence.

Corollary 2.2. *Let $\mathcal{W} \subset \mathcal{P}(\mathbb{R})$ denote the collection of all countable subsets of $\mathbb{R}$. Then* $\text{card}(\mathcal{W}) = c$.

Proof. Observe we have a bijection $\mathcal{W} \cong \mathbb{R}^{\mathbb{N}}$. □

We next consider some special cases of the problem mentioned in the introduction.

Example 2.3. We show the cardinality of the set of nowhere continuous functions on $\mathbb{R}$ is 2^c. In our notation,

$$\text{card}\left(\mathcal{C}_{\mathbb{R}}^{-1}(\{\emptyset\})\right) = 2^c.$$

The proof is a simple modification of Dirichlet's function. Given a function $\epsilon\colon \mathbb{R} - \mathbb{Q} \to \{1, 2\}$ define

$$f_\epsilon(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ \epsilon(x) & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

The assignment $\epsilon \mapsto f_\epsilon$ defines an injection $\{1, 2\}^{\mathbb{R} - \mathbb{Q}} \to \mathcal{C}_{\mathbb{R}}^{-1}(\{\emptyset\})$. The result now follows from Theorem 2.1.

At the other extreme is the well-known fact that the set of continuous functions on $\mathbb{R}$ has cardinality c.

Example 2.4. We have

$$\text{card}\left(\mathcal{C}_{\mathbb{R}}^{-1}(\{\mathbb{R}\})\right) = c.$$

For recall a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is determined by its values on the rationals. The restriction map $\text{Res}\colon \mathcal{C}_{\mathbb{R}}^{-1}(\{\mathbb{R}\}) \to \mathbb{R}^{\mathbb{Q}}$ with $\text{Res}(f) = f_{|\mathbb{Q}}$ is thus injective and the result follows from Theorem 2.1.

We next recall some standard terminology from the theory of metric spaces. Let X, d be a metric space. (We will omit mention of the metric d unless it is needed.) A subset $A \subseteq X$ is *dense* in X if $\overline{A} = X$ where $\overline{A}$ denotes the closure of A. The metric space X is *separable* if X contains a countable dense subset. The space X is *complete* if every Cauchy sequence in X is convergent. Finally, a subset $A \subseteq X$ is *nowhere dense* if $\overline{A}$ does not contain any nonempty open subsets of X. We have the following famous result (c.f. [5, Theorem 7.2], [6, Theorem 12.D]):

Theorem 2.5. (The Baire Category Theorem) *Let X be a complete metric space and $A_1, A_2, \ldots$ any sequence of nowhere dense subsets of X. Then*

$$X - \bigcup_{i=n}^{\infty} A_n \neq \emptyset.$$

□

We may extend Example 2.4 to separable metric spaces as follows:

Theorem 2.6. *Let X be a separable metric space and $A \subseteq X$ satisfying $X - A$ is countable and $\mathcal{C}_X^{-1}(\{A\}) \neq \emptyset$. Then*

$$\operatorname{card}\big(\mathcal{C}_X^{-1}(\{A\})\big) = c.$$

Proof. Observe the cardinality is at least c since adding any constant to $f \in \mathcal{C}_X^{-1}(\{A\})$ gives another function in this preimage. For the reverse inequality, we first show A contains a countable dense subset $Q \subseteq A$. To see this, take a countable subset Q' dense in X and consider the open balls centered at the elements of Q' with radius $\frac{1}{n}$ for all $n \in \mathbb{N}$. It is easy to see that these opens sets give a basis for the metric topology of X. Taking a point in each nontrivial intersection of one of these open balls with A yields the needed subset Q. As above then, a function f which is continuous on A is determined on A by its values on Q. This means the restriction map

$$\text{Res}\colon \mathcal{C}_X^{-1}(\{A\}) \to \mathbb{R}^{Q \cup (X-A)}$$

is an injection. Since $Q \cup (X - A)$ is countable by hypothesis, the result follows from Theorem 2.1. □

To illustrate the remaining possibility, we would like an example of a subset A of a metric space X with $\mathcal{C}_X^{-1}(\{A\}) = \emptyset$. Such an example is furnished, for $X = \mathbb{R}$, by taking $A = \mathbb{Q}$ ([1, Section 4.7]). More generally, this holds for any countable dense subset Q of a complete metric space X. To prove this, we need the the following important fact.

Theorem 2.7. *Let X, d be a metric space. Given any function $f\colon X \to \mathbb{R}$, the set C_f is a G_δ-set of X.*

Proof. For each $n = 1, 2, \ldots$ let

$$O_n = \{x \in X \mid \exists\, \delta > 0 \ \text{ such that both } \ d(x,s), d(x,t) < \delta \Rightarrow d(f(s), f(t)) < 1/n\}.$$

It is easy to check that O_n is an open subset of X and that $C_f = \bigcap_{n=1}^{\infty} O_n$. □

Example 2.8. Let Q be a countable dense subset of a complete metric space X. We show

$$\mathcal{C}_X^{-1}(\{Q\}) = \emptyset.$$

By Theorem 2.7, is suffices to show Q is not a G_δ-set. Suppose $Q = \bigcap_{n=1}^{\infty} O_n$ is an intersection of countably many open sets. Let $F_n = X - O_n$ and write $Q = \{q_1, q_2, \ldots\}$. Then we have $X = \bigcup_{n=1}^{\infty} F_n \cup \bigcup_{n=1}^{\infty} \{q_n\}$ is a countable union of nowhere dense sets, contradicting Theorem 2.5.

3. Extending Young's Construction

Given an arbitrary G_δ-set $A \subseteq \mathbb{R}$, Young constructed a function $f\colon \mathbb{R} \to \mathbb{R}$ with $C_f = A$. He thus proved the converse to Theorem 2.7 in the case $X = \mathbb{R}$, characterizing continuity sets as G_δ-sets. Young's construction is described in [3, pgs. 313-316]. An updated and more concise version of the construction appears in [2, pg. 30]. In [4], Kim extended Young's characterization of continuity sets to the general case of functions $f\colon X \to \mathbb{R}$ for X a metric space.

We give a version of Young's construction in a case closely related to the original, namely when X is a complete, separable metric space. Recall a point $x \in X$ is *isolated* if $\{x\}$ is an open subset of X. We need one nice fact about these metric spaces:

Theorem 3.1. *Let X be a complete, separable metric space without isolated points. Then X is either a single point or every nonempty open set in X has cardinality c.*

Proof. Assume X has more than one point and let $U \subseteq X$ be a nonempty open subset. Then $\overline{U}$ is itself a complete metric space without isolated points. The latter condition implies singleton sets $\{x\}$ for $x \in \overline{U}$ are nowhere dense in X. Write

$$\overline{U} = (\overline{U} - U) \cup \bigcup_{x \in U} \{x\}.$$

If U is countable we have written $\overline{U}$ as a countable union of nowhere dense sets which contradicts the Baire Category Theorem (Theorem 2.5).

On the other hand, to show $\mathrm{card}(X) \leq c$, let Q denote a countable dense subset of X. Let $\mathcal{CS} \subset Q^{\mathbb{N}}$ denote the set of all convergent sequences of elements of Q. Given a convergent sequence $(q_j) \in \mathcal{CS}$ define $L(q_j) = \lim_{j \to \infty} q_j \in X$. The assignment L gives a surjection $L \colon \mathcal{CS} \to X$ and the result follows from Corollary 2.2 □

We have the following immediate consequence:

Corollary 3.2. *Let X be a complete, separable metric space with more than one point and no isolated points. Let Q be a countable dense subset of X. Then $X - Q$ has cardinality c and is also dense in X.* □

Now let X be any complete, separable metric space and let I denote the set of isolated points of X. Let $A \subseteq X$ be a G_δ-set and suppose $I \subseteq A$. We construct a function $f \colon X \to \mathbb{R}$ with $C_f = A$. In fact, we wish to relate the cardinality of the set of such functions with that of the complement $X - A$. To make the connection, when $X - A$ is nonempty we take a function $\epsilon \colon X - A \to \{\frac{1}{3}, \frac{1}{2}\}$.

Write $A = \bigcap_{n=1}^{\infty} O_n$ for open sets $O_n \subseteq X$. We may assume the O_n are decreasing, i.e. $O_1 \supseteq O_2 \supseteq \cdots$. For if not, we rename $O_2 := O_1 \cap O_2$, $O_3 := O_1 \cap O_2 \cap O_3$, and so on. We let $O_0 = X$. For each $x \in X - A$, there is then a unique number $n(x) \in \{0, 1, 2, \ldots\}$ such that $x \in O_{n(x)} - O_{n(x)+1}$. Define

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in A \\ \epsilon(x)^{n(x)} & \text{if } x \in Q \cap (X - A) \\ -\epsilon(x)^{n(x)} & \text{if } x \in (X - Q) \cap (X - A) \end{cases}$$

We prove

Theorem 3.3. *Let X be a complete, separable metric space. Given a G_δ-set $A \subseteq X$ containing the isolated points of X, the function $f_\epsilon \colon X \to \mathbb{R}$ constructed above satisfies $C_{f_\epsilon} = A$.*

Proof. We first show f_ϵ is continuous on A. Suppose we have a convergent sequence $x_k \to x \in A$. Given $\varepsilon > 0$ choose n so large that $\frac{1}{2^n} < \varepsilon$. Since $x \in A = \bigcap_{n=1}^{\infty} O_n$ we can choose K so large that $k \geq K$ implies $x_k \in U_n$. Note that, for all such k, $f(x_k) \leq \frac{1}{2^n}$.

Suppose x is in the interior of $X - A$. Let $\varepsilon = |f(x)| = \epsilon(x)^{n(x)} > 0$. By Corollary 3.2 every open set containing x contains points y with $f(y)$ of opposite sign from $f(x)$ and thus with $|f(x) - f(y)| > \varepsilon$. On the other hand, if x is in the complement of $\overline{A} - A$ then $f(x) = \pm\epsilon(x)^{n(x)} \neq 0$ but $x = \lim_{k\to\infty} x_k$ is the limit of a sequence (x_k) of elements of A which, as such, satisfy $f(x_k) = 0$. □

Corollary 3.4. *Let X be a complete, separable metric space. Given a G_δ-set $A \subseteq X$ containing the isolated points of X, the assignment*

$$\epsilon \mapsto f_\epsilon : \left\{\frac{1}{3}, \frac{1}{2}\right\}^{X-A} \longrightarrow \mathcal{C}_X^{-1}(\{A\})$$

is injective. □

We can now deduce the complete calculation.

Theorem 3.5. *Let X be a complete, separable metric space. Let $I \subseteq X$ denote the set of isolated points of X. Let $\mathcal{A} \subseteq \mathcal{P}(X)$. Then*

$$\operatorname{card}(\mathcal{C}_X^{-1}(\mathcal{A})) = \begin{cases} 0 & \mathcal{A} \text{ does not contain a } G_\delta\text{-set containing } I \\ 2^c & \mathcal{A} \text{ contains a } G_\delta\text{-set } A \text{ with } I \subseteq A \text{ and } X - A \text{ uncountable} \\ c & \text{otherwise} \end{cases}$$

Proof. First observe that every function $f\colon X \to \mathbb{R}$ is trivially continuous at each of the isolated points of X. Thus if $I \not\subseteq A$ then $\mathcal{C}_X^{-1}(\{A\}) = \emptyset$. The result now follows from Theorem 2.6 and Corollary 3.4 □

Remarks 3.6. (1) While many of the results above generalize to the setting of real functions on an arbitrary metric space X, we cannot expect a succinct formula for the cardinalities of preimages of $\mathcal{C}_X\colon \mathbb{R}^X \to \mathcal{P}(X)$ at this level of generality. The reason is the cardinality of X need not be constant across different neighborhoods of X as is the case with X a separable complete metric space. For a simple example, take the space $X = (-\infty, 0) \cup (\mathbb{Q} \cap ([0, \infty))$ with metric inherited from $\mathbb{R}$. Then X has both countable and uncountable open intervals. Thus the location of a subset $A \subseteq X$ plays a role in the the number of functions with $C_f = A$.
(2) In the absence of the continuum hypothesis, we must adjust our conclusion in Theorem 3.5. Specifically, we must replace 2^c with 2^α where

$$\alpha = \sup\{\operatorname{card}(X - A) | \, A \in \mathcal{A} \text{ is a } G_\delta\text{-set containing the isolated points of } X\}$$

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About the authors:

Jiaming Chen is a junior mathematics major at Saint Joseph's University. This paper grew out of an honors project he undertook in a real analysis course taught by the second author.

Sam Smith received his Ph.D. in 1993 from the University of Minnesota. His research focuses on the homotopy theory of function spaces. Sam has been teaching at Saint Joseph's University since 1995. He currently enjoys directing the SJU Honors Program.

Department of Mathematics
Saint Joseph's University
Philadelphia, PA 19131
Email: jiaming.chen@sju.edu
Email: smith@sju.edu