

# INNOVATIVE POSSIBILITIES FOR UNDERGRADUATE TOPOLOGY

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## 1. INTRODUCTION.

The development of topology ranks as one of the great success stories of twentieth century mathematics. While the precise definition of a topological space is not yet a full century old, the subject has become a core requirement for virtually every branch of current mathematics research. From genetics to string theory to social science, applications of topology are diverse and pervasive. In its own right, topology is a vital and ever growing area, comprising dozens of subfields and engaging hundreds of researchers around the world.

The status of the undergraduate semester-course in topology is, unfortunately, not quite so glorious. Introductory topology tends to be viewed as a course suitable primarily for students headed to graduate school. While there are many superb textbooks in the field, most pitch the subject at an advanced level, including far more material than is possible to cover in one semester. Ironically, the axiomatic rigor that makes topology a model and

solid foundation for other fields is precisely the characteristic that makes it a difficult fit for the the undergraduate curriculum.

In this paper, I hope to indicate how an introductory topology course can become an accessible and popular elective for math majors of various strengths and diverse goals. One of the great advantages of topology is the almost visual elegance of its formalism. By emphasizing this quality, a teacher can help students to cope with the level of abstraction that is endemic to all theory courses. Of course, the subject matter of topology is its own greatest advertisement. By leading with the examples that have real geometric appeal, students can be motivated to tackle the more demanding aspects of the subjects. The structure of the course, moreover, should be sufficiently flexible to accommodate varied student needs. A topology course can function as a satisfying conclusion to course work in mathematics as well as a preliminary to graduate work. It can be the capstone of the pure mathematics curriculum and an entrée to independent research.

## **2. MOTIVATING THE ABSTRACTION.**

Opening a topology text to a random page illustrates a basic point about the subject. You are likely looking either at a very intriguing picture or at a page of pure formalism: theorems, lemmas, proofs. This is a basic dichotomy of the field. In my experience, students are drawn to the subject matter of topology. The challenge is to help students do the hard work of mastering the formalism. An excellent way to succeed is to show students how elegant and satisfying the formalism can be especially when applied to a concrete and familiar problem.

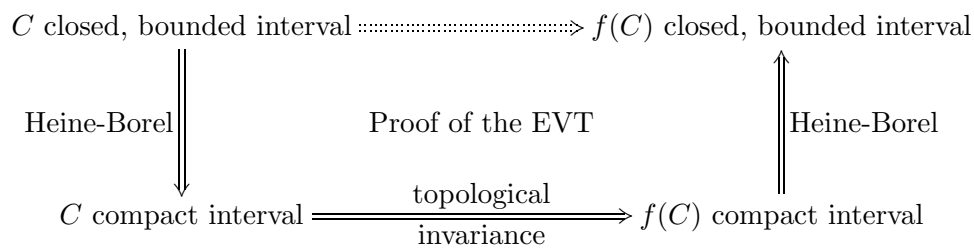
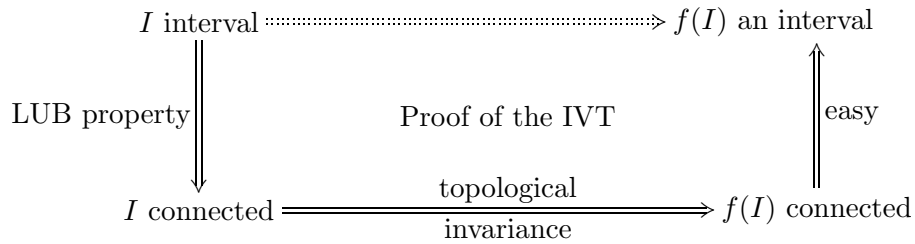
Perhaps the most important example of the power and intuitive appeal of the formalism of topology takes the students back to first semester calculus. Every math major has nodded in agreement to the picture proofs of the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT). Interestingly, these theorems are rarely proven in calculus. When teaching calculus, I try carefully to prove each step in the chain of implications  $\text{EVT} \Rightarrow \text{Local Extrema Theorem} \Rightarrow \text{Mean Value Theorem}$  and make a serious effort to explain how the Mean Value Theorem leads to the Fundamental Theorem of Calculus. But I never try to prove the IVT or EVT. The reason, of course, is that these proofs properly belong to topology.

In a topology course, the proofs of the IVT and EVT reveal the basic orthodoxy of the subject. They indicate the power of the first definitions: open sets, continuity and the crucial concept of topological invariance. The first step is to characterize the objects of the theorems, in this case intervals, in topological terms. This introduces the concepts *connectedness* and *compactness*. The hard step is to prove that the defined properties actually do characterize the objects, that is, that the connected sets in  $\mathbb{R}$  are precisely the intervals and the compact sets precisely the closed bounded intervals<sup>1</sup>. But now the power of the point of view takes over. By the definitions, these topological properties are preserved by continuous surjections, they are *invariants*. The proofs can, in fact, be visualized as “commutative”

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<sup>1</sup>As is often the case, the proof that the formalism really models the world is difficult!

diagrams:

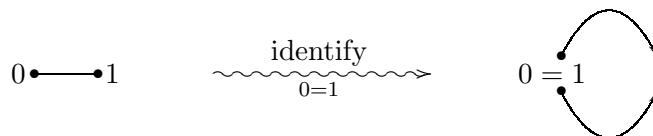


These two proofs make a compelling opening act to a course. For students intending to teach at the secondary level, the material is foundational. On the other hand, the notion of a topological invariant opens the door to advanced topics like the classification of spaces up to homeomorphism (the result  $\mathbb{R} \not\cong [a, b]$  is proved while results like  $S^n \not\cong \mathbb{R}^n$  make nice future goals). As with almost every topic in topology, there are many possible directions to pursue here and many intersections with other fields. Most importantly in terms of the course, these proofs can convince students of the elegance and the necessity of the formalism.

In a recent article in *The College Mathematics Journal* [1], Brenton and

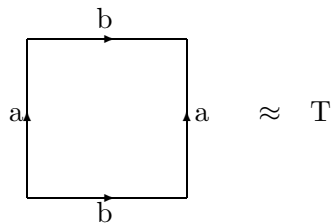
Edwards discuss how conceptual problems with sets become obstacles to understanding formal constructions like the quotient group in algebra. While students easily grasp the meaning of simple sets, e.g., sets of integers, they have problems with exotic sets like the set of cosets. Thus the quotient group in algebra remains mysterious due to the strange nature of its elements.

Topology is an excellent arena to work on tearing down this “cognitive barrier.” For example, consider, as the authors do, the case of  $\mathbb{Z}_3$ . Students are happy with the representation  $\{0, 1, 2\}$  but understandably less so with the coset representation  $\{0 + \mathbb{Z}, 1 + \mathbb{Z}, 2 + \mathbb{Z}\}$ . Consider the possible representations of the unit circle  $S^1$  : algebraic, trigonometric, geometric, or as the identification space  $[0, 1]/\{0 \sim 1\}$ . Which is the most natural? The last option, being quite visual, is not so intimidating.

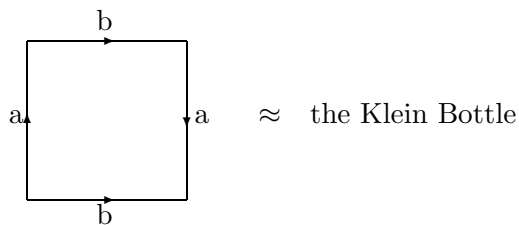


This picture introduces the quotient space, which is a good starting place for understanding more exotic sets. With some work, the topological isomorphism  $S^1 \cong \mathbb{R}/\mathbb{Z}$  can be understood. The task is not any easier than in algebra but it might be, for some students, better motivated.

Next consider the torus  $T$ . The three-dimensional representation of  $T$  is manageable (and a good flashback to multivariable calculus). But the identification space representation below is perhaps even more natural.



Routine exercises show that the topology is as expected. The product topology enters here via the homeomorphism  $T \approx S^1 \times S^1$ . This last fact, in turn, opens the door to topological groups. (Which spaces have this structure, how could we get an invariant, etc.?) Moreover, it is now a simple matter to obtain something strikingly different



The abstraction now has become as much philosophical as mathematical. It is pretty clear what object we intend by the diagram. But what can we say about its existence? It can be represented (immersed) in three-dimensional space but it doesn't truly live (embed) there. To understand the Klein bottle mathematically we must use the formalism. The good news is that the formalism – in this case, the quotient topology – is no more difficult than for the torus!

I believe that a topology course can feature the intuitive aspects of the subject without sacrificing the essential content. The preceding discussion hopefully serves to demonstrate how topics with concrete appeal can be used to introduce the key concepts of the course, providing a well-motivated

path to the standard material. In the next section, I will offer some specific suggestions on how to organize such a course.

### **3. STRUCTURING A COURSE.**

My basic goal in designing a topology course is to maximize the extent to which students discover the ideas of the course for themselves. Thus I emphasize problem sets over tests and in-class group work and student presentations over lectures. The advantage of this approach is clear: students feel ownership of ideas they have worked through themselves. The principal drawback is that less material can be covered than in a standard course which might disadvantage the very best students. However, the structure of the course described below is flexible enough to allow the most capable students to work on suitably challenging problems without completely overwhelming the less advanced students. The key to this flexibility lies in the role of homework in the course.

#### **3.1. PROBLEM SETS.**

Working problems is, of course, essential for learning mathematics. I recommend making homework problems central to the course in terms of both weight and focus. In my experience, students feel less intimidated if homework comprises a substantial fraction of the total points as they have more control over their score on problem sets than on a test. Of course, giving significant weight to homework also justifies assigning many problems!

I divide homework problems into two types: *practice problems*, which reinforce the basic concepts and are essentially routine, and *challenge problems*, whose resolution will be fundamentally more involved. Practice prob-

lems function as traditional homework problems – they have a due date and are graded and returned. Students are expected to attempt all of the practice problems.

Challenge problems, on the other hand, are elective: students can attempt those that interest them. Challenge problems have no due date but remain *open* until a correct solution has been presented to the class (see §3.3 below).<sup>2</sup> Thus the challenge problems give the course the atmosphere of a research seminar. Moreover, the division of problems gives the students choice about what topics they study in-depth without sacrificing basic common knowledge. One of the pleasures of teaching topology is the great variety of possible topics and problems at all levels of difficulty. The following examples indicate some possible ways the problem sets can be used to engage students in the course and shed new light on other areas of mathematics.

Homework problems should be treated as a focal point of the class, not just as subsidiary exercises yielding results with little bearing on the theory developed in lecture. For example, consider the standard material on closure, interior and boundary. A lecture on examples in Euclidean space motivates the definitions. Routine problems like  $\overline{A} = A^\circ \cup \partial(A)$  can be divided up between lecture and the practice problems. Practice problems can also be formulated as “determine the closure and interior” of subsets of interesting topological spaces. Such problems give students early hands-on experience with exotic topologies. Challenge problems can be harder standard results such as  $\overline{A \times B} = \overline{A} \times \overline{B}$  or (more challenging)  $\overline{\prod A_i} = \prod \overline{A_i}$  or

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<sup>2</sup>Since it is not possible to have all challenge problems presented in class, I also close challenge problems when they have been solved by a substantial number of students.

(very challenging) the Kuratowski 14-set problem [3, Exercise 20, p. 101].

Problem sets can be thematic. For example, a problem set on the Cantor set  $C$  is always popular with students. Practice problems can include the standard results ( $C$  is totally disconnected, perfect, etc.). Proving  $C \approx \{0, 1\}^\omega$  is also not difficult and makes a good exercise in understanding the product topology. Challenge problems could include (the rather strange result)  $C \approx C^\omega$  and (with hints) the various uniqueness results for  $C$ . This type of problem set can also open the door to independent research problems (see §4).

Homework topics can provide an instructive interaction with other areas of the undergraduate curriculum. An obvious example is topological groups. Here a couple of lectures can provide the background while practice problems can include proving properties like Hausdorff (assuming  $T_1$ ) and homogeneity of coset spaces (a useful exercise with the quotient topology). Challenge problems can include regularity of  $G$  and of  $G/H$  or, for a concrete example, proving  $GL(n, \mathbb{R})$  has exactly two path components.

The various incarnations of set theory in topology can be organized to provide a novel tour of Cantor's theory. For example, the space  $\overline{S_\Omega}$  (in Munkres' notation) consisting of all countable ordinals union the first uncountable ordinal  $\Omega$  in the order topology is a rich source of counterexamples. Studying properties of this space in a problem set provides an opportunity to review and discuss the well-ordering theorem, the construction of the ordinals and the continuum hypothesis. Other possibilities in this direction are the theory of Baire spaces and the set theory involved in proving the Tychonoff theorem.

### **3.2. IN-CLASS ASSIGNMENTS.**

Topology represents the context for one of the most famous pedagogical innovations of the twentieth century. The Moore method, as developed by R.L. Moore at the University of Texas, eliminates all texts and references from the course forcing the students to truly discover the results of the subject for themselves. While the Moore method is probably not suitable for most undergraduate classes, it is possible to recreate part of the experience for students using in-class assignments. I set aside several classes each semester, timed to coincide with the start of a new topic, during which the students split into groups and work on problems. For example, after introducing the separation axioms (Hausdorff, regularity, normality etc) the questions which axioms are *hereditary* (inherited by subspaces) and which *productive* (passed to products) are natural and open ended. Resolving these questions is easy in the case of the Hausdorff axiom, hard but possible for regularity and very difficult for normality. Thus these exercises give students a taste of two aspects of mathematical research: the consideration of propositions whose status is not known ahead of time and the great range of difficulty that similar-sounding statements can have. An alternate approach is to give each group a different, related problem (e.g. the many interrelationships among the countability and separation axioms) and have a volunteer from each group present its findings. In-class group work provides a lively alternative to lecture and an excellent opportunity for students to work on their topology “language skills” with help readily available.

### **3.3. STUDENT PRESENTATIONS.**

Every undergraduate mathematics major benefits from practice presenting

mathematics to their peers. Unfortunately, student presentations are generally an inefficient use of class time since students tend to be a passive audience for their peers. Challenge problem presentations provide a partial solution to this problem. When a student volunteers to present a challenge problem the status of the problem is still open. Since the proposed solution has not yet been graded, the students themselves must judge the correctness of the presentation as well as understand the techniques used. I award points both to the presenter and to the audience. If the proof presented is incorrect then an audience member can score points for pointing out difficulties or, even better, for finding a fix to a problem. Since many students have attempted each challenge problem the students have real incentive to be an active audience. In this context, topology offers an advantage over other courses in that there are not only proofs to present but challenging yet accessible constructions, as well. For instance, constructing an example of a path connected space which is nowhere locally connected makes for a nice challenge problem and presentation.

#### **4. INDEPENDENT RESEARCH DIRECTIONS.**

An introductory course in topology is an excellent spring-board to undergraduate research. The breadth and pervasiveness of the field makes it easy to design independent study projects which exhibit a strong interplay of topology with other areas of the undergraduate curriculum. Such integrated projects have a dual benefit: they appeal to students whose ultimate interests lie outside topology proper and they offer a vista onto the world of mathematical research where there are no fixed boundaries between fields.

Below are some possible directions for independent study and research in topology arranged roughly by their relationship to other (undergraduate-level) disciplines.

#### **4.1. ANALYSIS.**

After introductory courses in analysis and topology, there are a wealth of topics that can be pursued as independent studies. Examples include the theory of curves (Peano spaces and the Hahn-Mazurkiewicz Theorem), dimension theory and fractals. Elementary functional analysis involves topology, analysis, linear algebra, as well as some elementary ring theory. (See [7] for an approach emphasizing the topological aspects of function spaces.) For students intrigued by the Cantor set, there are interesting advanced results related to its universal properties and its many generalizations.

#### **4.2. GEOMETRY.**

The classification of compact surfaces is a beautiful and accessible theorem with real geometric appeal. (See [3,4].) The notions of Euler characteristic, genus and orientability are all illustrated by the theorem and can be pursued in more classical geometric contexts. Knot theory combines geometric appeal with connections to algebra and even physics. Geometric topics also occur in the theory of manifolds and in elementary differential topology.

#### **4.3. SET THEORY AND FOUNDATIONS.**

In addition to its many exotic set-theoretic examples (the ordinal space, the Tychonoff plank), advanced point-set topology offers a large supply of interesting problems in the intersection of topology and set theory. Consider, for example, the mathematics surrounding *Dowker's conjecture*. Since normality is not productive, it is natural to introduce stronger generalizations

which are. The class of *binormal* spaces is characterized by the fact that the product with the unit interval  $X \times I$  is normal for  $X$  binormal. The usual problem of finding a distinguishing example leads, in turn, to *Dowker's conjecture*: that, in fact,  $X \times I$  is normal for every normal space  $X$ . M.E. Rudin proved that Dowker's conjecture is unprovable using the axioms of set theory but the question of independence remains open. (See [5] for a relatively recent reference on current research.) An independent research project on Dowker's conjecture thus involves both interesting advanced topology and modern set theory.

#### **4.4. COMBINATORICS.**

For students interested in combinatorics, there are many possible projects with topological flavor. Compactifications are used in Ramsey theory, surfaces in graph theory and simplicial complexes in the theory of order. An introduction to the theory of arrangements makes a great research project with a nice interplay of topology and combinatorics.

#### **4.5. ALGEBRA.**

The first elements of algebraic topology – homotopy, the fundamental group, covering spaces – make a natural sequel to an introductory topology course. Advanced topics include the Galois correspondence between the subgroup lattice of  $\pi_1(X)$  and covering spaces of  $X$ , free groups via the fundamental group and the Seifert-Van Kampen Theorem. Other topics in the intersection of algebra and topology include topological groups, transformation groups and topological rings. Determining the structure of the various equivalence groups (homeomorphism, homotopy self-equivalence, etc.) of a topological space represents a fundamental and difficult problem in topol-

ogy. However, aspects of this problem can be pursued as undergraduate research.

#### **4.6. COMPUTER SCIENCE.**

Effective computation in topology and its relationship to functional programming is an area of active current research. For example, in [5] the authors provide an advanced but readable account of a program named *Kenzo* which gives a solution to the computability problem for the first homotopy groups of a simplicial complex. A student knowledgeable about topology and programming could implement an algorithm such as this in some special cases.

### **5. CONCLUSION.**

As regards the undergraduate curriculum, topology suffers from an embarrassment of riches. There are far too many wonderful problems, examples, theorems and applications for a single semester course. A topology course can be organized, however, to turn this breadth into an advantage, allowing students of varied interests and abilities to pursue different problems and paths. Moreover, the visual appeal and elegance of the subject can be used to help students master the considerable level of abstraction. Finally, topology offers a wealth of possibilities for capstone experiences and integrated undergraduate research.

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## **7. BRIEF BIOGRAPHY**

Sam Smith is an algebraic topologist and an associate professor of mathematics at Saint Joseph’s University in Philadelphia. Sam received his PhD from the University of Minnesota in 1993.