

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 19/2009

Homotopy Theory of Function Spaces and Related Topics

Organised by
Yves Félix (Louvain-la-Neuve)
Gregory Lupton (Cleveland)
Samuel Smith (Philadelphia)

April 5th – April 11th, 2009

ABSTRACT. This workshop brought together researchers studying a variety of problems related to the homotopy theory of function spaces. Topics covered included: evaluation maps and Gottlieb groups, the classification of gauge groups and of other function space components, algebraic models for function spaces both in the rational and in the p -local settings, operads, configuration spaces, free and based loop spaces and infinite-dimensional Lie groups.

Mathematics Subject Classification (2000): 55P48, 55P60, 55P62, 22E65.

Introduction by the Organisers

The study of function spaces from an algebraic topological point of view dates back, at least, to the 1950s. G. Whitehead posed the basic problem of classifying the path components of a function space up to homotopy type and obtained the first results on this problem as an early application of the Whitehead product. Subsequent work of Thom and Federer paved the way for the computation of algebraic invariants of function spaces.

In the late 1960s, Gottlieb initiated the study of the evaluation map, the evaluation subgroups and, in particular, the Gottlieb groups of space. In the 1970s, Hansen, Möller, Sutherland and others studied the homotopy classification problem for the components of a function space with many complete results. An early, famous application of Sullivan's rational homotopy theory, the Vigué-Sullivan model for the free loop space of a manifold, led to a solution of the closed geodesic problem and showed the power of Sullivan's algebraic models for homotopy theory.

The 1980s saw steady progress on function spaces, especially in the local settings. Following Sullivan's sketch, Haefliger described a model for the rational homotopy type of the space of sections of a nilpotent fibration. Félix, Halperin

and Thomas obtained global results on the vanishing and dimension of rationalized Gottlieb groups. Brown, Peterson and L. Smith gave a second rational model for function spaces in terms of Lannes's division functor. Finally and notably, Miller published his celebrated proof of the Sullivan conjecture concerning the contractibility of certain function spaces during this period, a major advance in homotopy theory.

In recent years, the study of function spaces and related topics has expanded and accelerated. Whitehead's original classification problem is actively researched in the context of gauge groups. Gottlieb groups remain a challenging computation problem in the integral setting and, after rationalization, are the subject of a basic conjecture in rational homotopy theory. The study of the free loop space of a manifold has undergone a renaissance with the discovery of Chas-Sullivan string topology. Meanwhile, the further development of algebraic models for the rational and p -local homotopy theory of function spaces has opened the field to whole new types of questions, computations and, significantly, applications of function space techniques in other areas of homotopy theory.

This workshop included 23 mathematicians with expertise and active research programs in these various areas. In addition to specialized talks, there were several invited survey talks on broad topics including Gottlieb groups, gauge groups and algebraic models for function spaces after localization. There were two extended problem sessions.

Workshop: Homotopy Theory of Function Spaces and Related Topics

Table of Contents

Méadhbh Boyle	
<i>An algebraic model for the homology of pointed mapping spaces out of a closed surface</i>	5
David Chataur	
<i>Division functors and mapping spaces</i>	7
Jean-Baptiste Gatsinzi (joint with Rugare Kwashira)	
<i>Rational homotopy groups of function spaces</i>	8
Marek Golasiński (joint with Daciberg Lima Gonçalves, Juno Mukai and Peter Wong)	
<i>Fox and Gottlieb homotopy groups and Whitehead products</i>	9
Daniel Henry Gottlieb	
<i>Coincidence numbers and the fundamental group</i>	12
Katsuhiko Kuribayashi	
<i>Models for function spaces and applications</i>	13
Andrey Lazarev (joint with J. Chuang)	
<i>Maurer-Cartan moduli spaces and associated characteristic classes</i>	16
Paolo Salvatore	
<i>Cyclic formality of the framed little discs operad</i>	18
Jeff Strom	
<i>Miller spaces</i>	18
Svjetlana Terzić (joint with Jelena Grbić)	
<i>Integral loop homology of complete flag manifolds</i>	20
Shuichi Tsukuda (joint with Yasuhiko Kamiyama)	
<i>On the configuration spaces of a certain n-arms machine in the Euclidean space</i>	22
Antonio Viruel	
<i>Equivalences of a product and Mal'cev quasirings</i>	23
Christoph Wockel	
<i>Non-integral central extensions of loop groups via gerbes</i>	24

Abstracts

An algebraic model for the homology of pointed mapping spaces out of a closed surface

MÉADHBH BOYLE

Let X be a 2-connected pointed topological space which has the homotopy type of a CW-complex or a simplicial set. Let S_g denote a closed surface of genus g with a choice of base point. Consider the pointed mapping space from S_g into X , denoted $\text{Map}_*(S_g, X)$. We construct a model for the homology of $\text{Map}_*(S_g, X)$, under certain conditions.

The mapping space appears in the fibration

$$\Omega^2 X \rightarrow \text{Map}_*(S_g, X) \rightarrow (\Omega X)^{2g}$$

and the construction which we give mimics this fibration on an algebraic level. When X is a H -space then the algebraic structure turns out to be quite straightforward. However, when X is not specifically a H -space then the construction requires using algebraic models for the chains on ΩX and $\Omega^2 X$.

An Alexander-Whitney coalgebra is a coalgebra, C , whose diagonal is itself a coalgebra map up to strong homotopy. Alexander-Whitney coalgebras were first introduced and studied by Hess, Parent, Scott and Tonks in [7] and they occur naturally in topology by taking the normalized chains on a simplicial set. We take an Alexander-Whitney coalgebra C as a model for C_*X . The cobar construction, denoted $\underline{\Omega}C$, is taken as a model for the chains on ΩX .

When C is an Alexander-Whitney coalgebra, there is a coalgebra structure on the cobar construction of C ([7]), and so the cobar construction can be applied to the cobar construction. As the cobar construction is a tensor algebra, applying the cobar construction to the cobar construction gives a very big space with a differential which is difficult to deal with. In 2007, Hess and Levi introduced a model for the chains on the double loop space of an Alexander-Whitney coalgebra, denoted \mathcal{L}_2C ([6]). Roughly speaking, the Hess-Levi model is defined as the cotensor product of $\underline{\Omega}(C \oplus \overline{C})$ and R over $\underline{\Omega}C$, where \overline{C} is the desuspension of C and R is the ground ring of C . This model has a differential defined on it and is generally much more manageable. We use the Hess-Levi model \mathcal{L}_2C as a model for the chains on $\Omega^2 X$.

Vital to our main calculation is the use of twisting cochains. Let Δ denote the diagonal map on a coalgebra C ; μ denote the multiplication map for an algebra A ; and d_A and d_C denote the differentials on A and C respectively. A twisting cochain is a degree -1 map $t: C \rightarrow A$ such that $d_A t + t d_C = \mu(t \otimes t)\Delta$. The twisted tensor product of C and A , denoted $C \otimes_t A$, is the graded module $C \otimes A$ with the differential defined on it being $d_C \otimes 1_A + 1_C \otimes d_A - (1_C \otimes \mu)(1_C \otimes t \otimes 1_A)(\Delta \otimes 1_A)$, [8].

The twisting cochain required is $\tau: (\underline{\Omega}C)^{\otimes 2g} \rightarrow \mathcal{L}_2C$. We construct a twisting cochain $t: \underline{\Omega}C$ and τ is defined as the composite $t \circ \Gamma$ where $\Gamma: (\underline{\Omega}C)^{\otimes 2g} \rightarrow \underline{\Omega}C$ is the commutator map.

We also use results of Eilenberg-Moore ([3, p.220, Thm 12.1]) and Gugenheim-Munkholm ([5, p.15, 2.2*]) for the main result in this paper.

Theorem 1.1. *Let X be a 2-connected topological space of the homotopy type of a CW-complex or a simplicial set. Let R be either an integral domain in which 2 is a unit or a field of characteristic 2. Let $C \cong C_*X$ be an Alexander-Whitney coalgebra over R such that $\underline{\Omega}C$ is primitively generated. If C is formal or a double suspension then*

$$H_*(\text{Map}_*(S_g, X)) \cong H_*(\mathcal{L}_2C \otimes_{\tau} (\underline{\Omega}C)^{\otimes 2g}),$$

where $\tau: (\underline{\Omega}C)^{\otimes 2g} \rightarrow \mathcal{L}_2C$ is a twisting cochain.

This model provides an explicit way of calculating the homology of the pointed mapping space under the given conditions. The cobar construction and the Hess-Levi construction both have explicit and relatively easily calculable differentials defined on them. The model presented here is therefore preferable to other methods as it is integral and easier to compute.

REFERENCES

- [1] J. F. Adams. "On the cobar construction". Proceedings of the National Academy of Science, USA 42 (1956) 409-412.
- [2] F. R. Cohen, J. C. Moore, J. A. Neisendorfer. "Torsion in homotopy groups". Ann. of Math. (2) 109 (1979) 121-168.
- [3] Samuel Eilenberg and John C. Moore. "Homology and Fibrations I: Coalgebras, Cotensor Product and its Derived Functors". Comment. Math. Helv. 40 (1966) 199-236.
- [4] Yves Felix, Stephen Halperin and Jean-Claude Thomas. *Rational Homotopy Theory*. Springer-Verlag, New York, 2001.
- [5] V. K. A. M. Gugenheim and H. J. Munkholm. "On the Extended Functoriality of Tor and Cotor". Journal of Pure and Applied Algebra 4 (1974) 9-29.
- [6] Kathryn Hess and Ran Levi. "An Algebraic Model for the Loop Space Homology of a Homotopy Fiber". Algebraic and Geometric Topology 7 (2007) 1699-1765.
- [7] K Hess, P-E Parent, J Scott, A Tonks. "A canonical enriched Adams-Hilton model for simplicial sets". Adv. Math. 207 (2006) 847-875.
- [8] Dale Husemoller, John C. Moore, James Stasheff. "Differential Homological Algebra and Homogeneous Spaces". Journal of Pure and Applied Algebra 5 (1974) 113-185.
- [9] May, Peter. *Simplicial Objects in Algebraic Topology*. D. Von Nostrand Company, Inc, London, 1967.
- [10] R. J. Milgram. "Iterated Loop Spaces". Ann. of Math. (2) 84 (1966) 386-403.
- [11] R. H. Szczarba. "The Homology of Twisted Cartesian Products". Trans. Amer. Math. Soc. 100 (1961) 197-216.
- [12] Spanier, Edwin H. *Algebraic Topology*. McGraw-Hill, Inc., London, 1966.
- [13] D. Tanré. "Homotopie rationnelle: modèles de Chen, Quillen, Sullivan". Lecture Notes in Mathematics 1025, Springer, Berlin (1983).

Division functors and mapping spaces

DAVID CHATAUR

Lannes's division functors for unstable algebras over the Steenrod algebra \mathcal{A}_p provides us with a left adjoint to the completed tensor product of two unstable algebras.

In fact as noticed by Bousfield, Peterson and Smith, Division functors can be defined for algebras over a triple T as soon as the tensor product of two T -algebras is again a T -algebra. They used this key fact to define a division functor into the category of commutative differential graded algebras over the field of rational numbers.

Let $A_{PL}(-)$ be the Sullivan's functor of PL -forms. When X is a connected, finite simplicial set and when Y is a connected, nilpotent simplicial set of finite type the division functor

$$(M(Y) : A_{PL}(X)),$$

where $M(Y)$ is a cofibrant CDGA replacement of $A_{PL}(Y)$, is a rational model of $map(X, Y)$. This is the derived division functor of

$$(A_{PL}(Y) : A_{PL}(X))$$

in the sens of Quillen's homotopical algebra.

In this talk we explain how to build a p -adic model for $map(X, Y)$. The construction is due to B. Fresse, it relies upon a construction of a division functor for homotopy commutative algebras. Homotopy commutative algebras or E_∞ -algebras are a particular type of algebras over an operad. We will explain how this multiplicative structure arises for singular cochains, how one can do homotopy theory and give algebraic models for spaces and then for mapping spaces in terms of derived division functor.

Finally we will explain some joint work with K. Kuribayashi where we constructed a spectral sequence for mapping spaces using division functors.

REFERENCES

- [1] A. K. Bousfield, C. Peterson and L. Smith, The rational homology of function spaces, Arch. Math. 52 (1989), 275-283
- [2] D. Chataur; K. Kuribayashi : An operadic model for a mapping space and its associated spectral sequence. J. Pure Appl. Algebra 210 (2007), no. 2, 321-342.
- [3] B. Fresse, Derived division functors and mapping spaces, preprint [arXiv:math.At/0208091](https://arxiv.org/abs/math/0208091) (2002).
- [4] M. Mandell, E_∞ algebras and p -adic homotopy theory. Topology 40 (2001), no. 1, 43-94.
- [5] J. E. McClure and J. H. Smith, Multivariable cochain operations and little n -cubes, J. Amer. Math. Soc. 16 (2003), 681-704.
- [6] L. Schwartz, Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture, Chicago Lectures in Mathematics, University of Chicago Press, 1994.

Rational homotopy groups of function spaces

JEAN-BAPTISTE GATSINZI

(joint work with Rugare Kwashira)

Throughout this paper, spaces are assumed to be 1-connected finite CW-complexes. Given a map between spaces $f : X \rightarrow Y$, we denote by $\text{map}(X, Y, f)$ (respectively $\text{map}_*(X, Y, f)$) the space of (respectively pointed) mappings from X into Y that are homotopic to f . We also (abusively) denote by $f : \mathbb{L}(V) \rightarrow \mathbb{L}(W)$ a Quillen model of f . Lupton and Smith [4] showed that from the Lie bracket of $\mathbb{L}(V)$ and $\mathbb{L}(W)$, one can extend the notion of derivation of a differential graded Lie algebra to a derivation with respect to a map of Lie algebras. A *f-derivation of degree n* is a linear map $\theta : \mathbb{L}(V) \rightarrow \mathbb{L}(W)$ that increases the degree by n and satisfies $\theta([x, y]) = [\theta(x), f(y)] + (-1)^{n|x|}[f(x), \theta(y)]$ for $x, y \in \mathbb{L}(V)$. Denote by $\text{Der}_n(\mathbb{L}(V), \mathbb{L}(W); f)$ the space of all f -derivations of degree n from $\mathbb{L}(V)$ to $\mathbb{L}(W)$. Define

$$D : \text{Der}_n(\mathbb{L}(V), \mathbb{L}(W); f) \rightarrow \text{Der}_{n-1}(\mathbb{L}(V), \mathbb{L}(W); f)$$

by $D(\theta) = \delta_W \theta - (-1)^{|\theta|} \theta \delta_V$. Then $(\text{Der}_*(\mathbb{L}(V), \mathbb{L}(W); f), D)$ is a differential graded vector space.

The adjoint map associated to f is the derivation

$$\text{ad}_f : \mathbb{L}(W) \rightarrow \text{Der}(\mathbb{L}(V), \mathbb{L}(W); f)$$

where $\text{ad}_f(w)(x) = [w, f(x)]$, $w \in \mathbb{L}(W)$.

They then proved the following vector space isomorphisms

$$\pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(\text{Der}(\mathbb{L}(V), \mathbb{L}(W); f)),$$

$$\pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(s\mathbb{L}(W) \oplus \text{Der}(\mathbb{L}(V), \mathbb{L}(W); f)).$$

For a connected differential graded algebra $T(V)$, there an acyclic differential $T(V)$ -module of the form $(T(V) \otimes (\mathbb{Q} \oplus sV), D)$ [1],[3]. Gatsinzi [2] proved that the map

$$\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)) \xrightarrow{\cong} s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$$

is an isomorphism of graded vector spaces. This can be extended for a map $f : \mathbb{L}(V) \rightarrow \mathbb{L}(W)$. The adjoint action of TW on $\mathbb{L}(W)$ combined with Uf induces a $T(V)$ -module structure on $\mathbb{L}(W)$. We have the following results.

Theorem 1.1. *There is a bijective map of differential graded vectors*

$$F : \text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W); f) \rightarrow s\mathbb{L}(W) \oplus \text{Der}(\mathbb{L}(V), \mathbb{L}(W); f).$$

Hence $\pi_*(\text{map}_*(X, Y, f)) \otimes \mathbb{Q} \cong \text{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(W))$.

Corollary 1.2. *If $\pi_*(Y) \otimes \mathbb{Q}$ is finite dimensional, then $\pi_*(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ and $\pi_*(\text{map}_*(X, Y; f))$ are both finite dimensional.*

Consider $\omega : \text{map}(X, Y, f) \rightarrow Y$ the evaluation at the base point. The generalized Gottlieb group is defined by $G_n(Y, X; f) = \text{im}(\pi_*(\omega))$. Following [2] define an evaluation map

$$\text{ev} : \text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)) \rightarrow \mathbb{L}(W)$$

by $\text{ev}(k) = k(1)$. Consider the induced map in homology

$$H_*(\text{ev}) : \text{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(W)) \rightarrow H_*(\mathbb{L}(W), \delta)$$

Proposition 1.3.

$$G_n(Y, X; f) \otimes \mathbb{Q} \cong \text{im}(H_*(\text{ev})).$$

REFERENCES

- [1] J.F. Adams, P.J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv., 30 (1956), 305-330.
- [2] J.-B. Gatsinzi, *The homotopy Lie algebra of classifying spaces*, J. Pure Appl. Algebra, 120 (1997), 281–289.
- [3] J.M. Lemaire, *Algèbres connexes et homologie des espaces de lacets*, Lecture Notes in Mathematics 422, Springer, Berlin, 1974.
- [4] G. Lupton, S.B. Smith, *Rationalized evaluation subgroups of a map II: Quillen models and adjoint maps*, J. Pure Appl. Algebra, 209, (2007), 173–188.

Fox and Gottlieb homotopy groups and Whitehead products

MAREK GOLASIŃSKI

(joint work with Daciberg Lima Gonçalves, Juno Mukai and Peter Wong)

1. Gottlieb groups. Let X be a connected pointed space. Recall that the k -th *Gottlieb group* $G_k(X)$ of X has been defined by Gottlieb in [8, 9] as the subgroup of the k -th homotopy group $\pi_k(X)$ consisting of all elements which can be represented by a map $f : \mathbb{S}^k \rightarrow X$ such that $f \vee \text{id}_X : \mathbb{S}^k \vee X \rightarrow X$ extends (up to homotopy) to a map $F : \mathbb{S}^k \times X \rightarrow X$.

Given $\alpha \in \pi_k(\mathbb{S}^n)$ for the n th sphere \mathbb{S}^n and $k \geq 1$, we deduce that $\alpha \in G_k(\mathbb{S}^n)$ if and only if $[\iota_n, \alpha] = 0$. In view of [6], we have a table of $G_{n+k}(\mathbb{S}^n)$ for $1 \leq k \leq 13$ and $2 \leq n \leq 26$:

Let now $\mathbb{F}P^n$ be the n -projective space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and put $d = \dim_{\mathbb{R}} \mathbb{F}$. Then, write $\gamma_n = \gamma_{n, \mathbb{F}} : \mathbb{S}^{(n+1)d-1} \rightarrow \mathbb{F}P^n$ for the quotient map. Denote by $i_{k, n, \mathbb{F}P} : \mathbb{F}P^k \hookrightarrow \mathbb{F}P^n$ for $k \leq n$ the inclusion map. Then, by [7], we have the following results.

Theorem 1.1. *The equality $G_{k+n}(\mathbb{R}P^n) = \gamma_{n*} G_{k+n}(\mathbb{S}^n)$ holds if $k \leq 7$ except the following pairs: $(k, n) = (3, 4), (4, 4), (5, 4), (6, 4), (5, 6), (7, 8), (7, 11), (3, 2^i - 3)$ with $i \geq 4$ and $(6, 2^i - 5)$ with $i \geq 5$.*

Furthermore,

$$(1) G_7(\mathbb{R}P^4) \supseteq 12\pi_7(\mathbb{R}P^4);$$

- (2) $G_{10}(\mathbb{R}P^4) \supseteq 3\pi_{10}(\mathbb{R}P^4)$;
- (3) $G_{11}(\mathbb{R}P^6) \supseteq 30\pi_{11}(\mathbb{R}P^6)$;
- (4) $G_{15}(\mathbb{R}P^8) \supseteq 2520\pi_{15}(\mathbb{R}P^8)$;
- (5) $G_{18}(\mathbb{R}P^{11}) \supseteq 2\pi_{18}(\mathbb{R}P^{11})$;
- (6) $G_{2^i}(\mathbb{R}P^{2^i-3}) \supseteq 2\pi_{2^i}(\mathbb{R}P^{2^i-3})$ for $i \geq 4$.

Theorem 1.2. (1) Let $k = 1, 2$. Then:

$$G_{k+2n+1}(\mathbb{C}P^n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \pi_{k+2n+1}(\mathbb{C}P^n) \cong \mathbb{Z}_2, & \text{if } n \text{ is odd.} \end{cases}$$

(2)

$$G_{2n+4}(\mathbb{C}P^n) \supseteq \begin{cases} (24, n)\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{24}{(24, n)}}, & \text{if } n \text{ is even;} \\ \frac{(24, n+3)}{2}\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{48}{(24, n+3)}}, & \text{if } n \geq \text{is odd.} \end{cases}$$

In particular, $G_{2n+4}(\mathbb{C}P^n) = 2\pi_{2n+4}(\mathbb{C}P^n)$ if $n \equiv 2, 10 \pmod{12} \geq 10$ except $n = 2^{i-1} - 2$ or $n \equiv 1, 17 \pmod{24} \geq 17$ and $G_{2n+4}(\mathbb{C}P^n) = \pi_{2n+4}(\mathbb{C}P^n)$ if $n \equiv 7, 11 \pmod{12}$.

$$(3) G_{2n+7}(\mathbb{C}P^n) = \pi_{2n+7}(\mathbb{C}P^n) \text{ if } n \equiv 2, 3 \pmod{4}.$$

Theorem 1.3. (1) $G_{4n+3}(\mathbb{H}P^n) \supseteq \frac{3+(-1)^{n+1}}{2}(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3})$;

$$(2) G_{4n+6}(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_{n*}\pi_{4n+6}(\mathbb{S}^{4n+3}) \cong \mathbb{Z}_{\frac{24}{(24, n+2)}} \text{ for } n \geq 2;$$

$$(3) G_k(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_n\nu_{4n+3} \circ \pi_k(\mathbb{S}^{4n+6}) \cong \mathbb{Z}_2.$$

2. Various evaluation groups. Using the modern language of homotopy theory, we reintroduce in [3] so-called torus homotopy groups considered in [2].

Let X be a pointed space. For $n \geq 1$, the n -th Fox group of X is defined to be

$$\tau_n(X) = [\Sigma(\mathbb{T}^{n-1} \cup *), X],$$

where \mathbb{T}^k denotes the k -dimensional torus and Σ the reduced suspension.

The obvious projection map $\mathbb{T}^{n-1} \cup * \rightarrow \mathbb{S}^{k-1}$ leads to imbeddings

$$\pi_k(X) \longrightarrow \tau_n(X)$$

for $1 \leq k \leq n$.

Given a space X , we define in [5] the *Gottlieb-Fox groups* to be the evaluation subgroups

$$G\tau_n := G\tau_n(X, x_0) := \text{Im}(ev_* : \tau_n(\text{Map}(X, X), \text{id}_X) \rightarrow \tau_n(X, x_0))$$

of the torus homotopy groups τ_n for $n \geq 1$.

Next, let G denote a finite group acting on a compactly generated Hausdorff path connected space X with a basepoint. F. Rhodes has introduced in [10] the notion of the fundamental group $\sigma(X, x_0, G)$ of the pair (X, G) , where x_0 is a basepoint in X . Then, F. Rhodes has defined in [11] higher groups $\sigma_n(X, x_0, G)$ of the pair (X, G) for $n \geq 1$ which is an extension of $\tau_n(X, x_0)$ by G so that

$$1 \rightarrow \tau_n(X, x_0) \rightarrow \sigma_n(X, x_0, G) \rightarrow G \rightarrow 1$$

is exact.

The evaluation subgroup

$$G\sigma_n := G\sigma_n(X, x_0, G) := \text{Im}(ev_* : \sigma_n(X^X, \text{id}_X, G) \rightarrow \sigma_n(X, x_0, G))$$

of σ_n is called the n -th *Gottlieb-Rhodes group* of a G -space X .

To relate the Gottlieb-Rhodes groups with the Gottlieb-Fox groups, we consider in [4] the homomorphism $p_n : G\sigma_n \rightarrow G$ given by $[f; g] \mapsto g$ for $[f; g] \in G\sigma_n$.

Theorem 2.1. *The following sequence*

$$(1) \quad 1 \rightarrow G\tau_n \rightarrow G\sigma_n \xrightarrow{p_n} G_0 \rightarrow 1$$

is exact. Here, G_0 is the subgroup of G consisting of elements g considered as homeomorphisms of X which are freely homotopic to id_X .

At the end, for any pointed spaces X and V , we define in [3] the *generalized Gottlieb group*

$$G(\Sigma V, X) = \text{Im}(ev_*[\Sigma V(X^X, \text{id}_X)] \rightarrow [\Sigma V, (X, x_0)])$$

and the *V-Fox group*

$$\tau_V(X) = [\Sigma(V \cup *), X].$$

Given a space W , the group $[\Sigma V, X]$ can be regarded as a subgroup of $\tau_{V \times W}(X)$ via the projection $V \times W \rightarrow V$. Then, we have shown in [4]:

Proposition 2.2. *The generalized Gottlieb group $G(\Sigma V, X)$, regarded as a subgroup of $\tau_{V \times W}(X)$ is central in $\tau_{V \times W}(X)$. In particular, it is central in $[\Sigma V, X]$.*

Now, given $\alpha \in [\Sigma V, X]$ and $\beta \in [\Sigma W, X]$, consider the generalized Whitehead product (see [1] for details)

$$\alpha \circ \beta : \Sigma(V \wedge W) \rightarrow X.$$

Then, the composite

$$\Sigma((V \times W \cup *) \rightarrow \Sigma(V \times W) \rightarrow \Sigma(V \wedge W) \xrightarrow{\alpha \circ \beta} X$$

determines an element in the $V \times W$ -Fox group $\tau_{V \times W}(X)$.

Theorem 2.3. (1) *Given $\alpha \in [\Sigma V, X]$ and $\beta \in [\Sigma W, X]$, the image of the generalized Whitehead product $\alpha \circ \beta$ in $\tau_{V \times W}(X)$ is the commutator $[\alpha, \beta]$ of α and β .*

(2) *If $\alpha \in G(\Sigma V, X)$ then the generalized Whitehead product $\alpha \circ \beta = 0$.*

We also point out that in [5] generalized Fox torus groups are used to unify different approach to generalized Jacobi identities.

REFERENCES

- [1] M. Arkowitz, *The generalized Whitehead product*, Pacific J. Math. 12 (1962), 7–23.
- [2] R. Fox, *Homotopy groups and torus homotopy groups*, Ann. of Math. (2) 49 (1948), 471–510.
- [3] M. Golasinski, D.L. Gonçalves, P. Wong, *Generalizations of Fox homotopy groups, Whitehead products and Gottlieb groups*, Ukrain. Mat. Zh. 57 (2005), no. 3, 320–328; translation in Ukrainian Math. J. 57 (2005), no. 3, 382–393.
- [4] ———, *Equivariant evaluation subgroups and Rhodes groups*, Cah. Topol. Géom. Différ. Catég. 48 (2007), no. 1, 55–69.
- [5] ———, *On Fox spaces and Jacobi identities*, Math. J. Okayama Univ. 50 (2008), 161–176.
- [6] M. Golasinski, J. Mukai, *Gottlieb groups of spheres*, Topology 47 (2008), no. 6, 399–430.
- [7] ———, *Gottlieb and Whitehead center groups of projective spaces*, in preparation.
- [8] D. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. of Math. 87 (1965), 840–856.
- [9] ———, *Evaluation subgroups of homotopy groups*, Amer. J. of Math. 91 (1969), 729–756.
- [10] F. Rhodes, *On the fundamental group of a transformation group*, Proc. London Math. Soc. (3) 16 (1966) 635–650.
- [11] ———, *Homotopy groups of transformation groups*, Canad. J. Math. 21 (1969), 1123–1136.

Coincidence numbers and the fundamental group

DANIEL HENRY GOTTLIEB

For M and N closed oriented connected smooth manifolds of the same dimension, we consider the mapping space $\text{map}(M, N; f)$ of continuous maps homotopic to $f : M \rightarrow N$. We will show that the evaluation map from the space of maps to the manifold N induces a nontrivial homomorphism on the fundamental group only if the self coincidence number of f , denoted $\Lambda_{f,f}$, equals zero. Since $\Lambda_{f,f}$ is equal to the product of the degree of f and the Euler–Poincaré number of N , we obtain results related to earlier results about the evaluation map and the Euler–Poincaré number, [1].

This result has already appeared in [2]. It was discovered in response to a conjecture of Dusa McDuff in [3]. It was proved using the coincidence index axioms of Chris Staecker in [5]. It generalizes a well known result of [1], for much more restrictive hypotheses. The result in [5] was generalized by John Stallings in [6]. This started a chain of increasingly generalized results in group cohomology. It led me to conjecture a version of the result in the first paragraph for group cohomology by generalizing the concept of the degree of a map from manifolds to topological spaces, which was shown to be literally false by T. Shick and A. Thom in [5]. But then they modified the definition of the degree, and proved the modified conjecture. Finally, they observed that the theorem they proved did not need the generalized concept of degree that was used to make the conjecture.

REFERENCES

- [1] Daniel Henry Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856

due to Haefliger [7], Brown and Szczarba [2], which are called the HBS models, and a model for the evaluation map are described with a few examples. Moreover, we will explain how the models are relevant to investigate a topological invariant for example, Gottlieb groups [6] and Kedra-McDuff μ -classes [10].

It is expected that ideas in [8], [11] and [3] are applicable when understanding topological invariants and notions algebraically. Indeed, in ongoing work [12], the rational visibility of a Lie group in the space of self-homotopy equivalences of a homogeneous space is investigated by means of tools developed in the previous papers.

We are convinced that, in adding the results which appear in origins of function space models [21], [22] and [7], explicit HBS models and derivations on Sullivan models used in [14], [15], [16] and [4] are useful tools for the study of function spaces, evaluation subgroups and other topological invariants.

My talk consists of the following subjects ¹.

- Construction of the HBS models with Lannes' division functor.
- An explicit component model for a function space and an rational model for the evaluation map.
- Applications: Gottlieb groups, Kedra-McDuff μ -classes.

1. APPLICATIONS

The HBS model for a function space and our explicit model for the evaluation map are applicable to the computation of a Gottlieb group and to the study of appropriate characteristic classes. We here describe computational examples.

Consider the S^1 -bundle $S^1 \rightarrow X_f \rightarrow T^n$ over the n -dimensional torus T^n with the classifying map f which is represented by $\rho_f = \sum_{i < j} c_{ij} t_i t_j$ in $H^2(T^n; \mathbb{Z}) \cong [T^n, K(\mathbb{Z}, 2)]$. Here $\{t_i\}_{1 \leq i \leq n}$ is a basis of $H^1(T^n; \mathbb{Z})$. Define an $(n \times n)$ -matrix A_f by $A_f = (c'_{ij})$, where $c'_{ij} = c_{ij}$ for $i < j$, $c'_{ij} = -c_{ji}$ for $i > j$ and $c_{ii} = 0$. We regard A_f as a matrix with entries in \mathbb{Q} . Then the rank of A_f is denoted by $\text{rank} A_f$. By analyzing our model for the evaluation map, the following theorem is established.

Theorem 1.1. [8] $G_1(X_f) \cong \mathbb{Z}^{\oplus(1+n-\text{rank} A_f)}$.

We next give μ -classes due to Kedra and McDuff [10] a description with the HBS model. In order to define the characteristic classes we first recall the coupling class. In what follows, we write $H^*(X)$ for the cohomology with coefficients in the rational field. Let M be a k -dimensional manifold. Consider the Leray-Serre spectral sequence $\{E_r, d_r\}$ for a fibration $M \xrightarrow{i} E \xrightarrow{\pi} B$ for which $\pi_1(B)$ act trivially on $H^k(M) = \mathbb{K}$. Let $\{F^p H^*\}_{p \geq 0}$ denote the filtration of $\{E_r, d_r\}$. Then the integration along the fibre (the cohomology push forward) $\pi! : H^{p+k}(E) \rightarrow H^p(B)$ is defined by the composite

$$H^{p+k}(E) = F^0 H^{p+k} = F^p H^{p+k} \rightarrow E_\infty^{p,k} \rightarrow \dots \rightarrow E_2^{p,k} \cong H^p(B; H^k(M)) \cong H^p(B).$$

¹We also refer the reader to [5] for an operadic model for a function space which is described in terms of p -adic homotopy theory while the topic will be not dealt with in the talk.

Let (M, a) be a $2m$ -dimensional c -symplectic manifold [13] and \mathcal{G} denote the monoid $\text{aut}_1(M)$ of self-homotopy equivalences. Let $M \xrightarrow{i} M_{\mathcal{G}} \xrightarrow{\pi} B\mathcal{G}$ be the universal M -fibration; see [18, Proposition 7.9].

Proposition 1.2. [9, Proposition 2.4.2] [10, Proposition 3.1] *Suppose that $H^1(M) = 0$, then the element $a \in H^2(M)$ is extendable to an element $\bar{a} \in H^2(M_{\mathcal{G}})$. Moreover, there exists a unique element $\tilde{a} \in H^2(M_{\mathcal{G}})$ that restricts to $a \in H^2(M)$ and such that $\pi!(\tilde{a}^{m+1}) = 0$. In fact the element \tilde{a} has the form*

$$\tilde{a} = \bar{a} - \frac{1}{n+1} \pi^* \pi!(\bar{a}^{m+1}).$$

The class \tilde{a} in Proposition 1.2 is called *the coupling class*.

Definition 1.3. [10, Section 3.1] [9, Section 2.4] We define $\mu_k \in H^{2k}(B\mathcal{G})$, which is called the k th μ -class, by $\mu_k := \pi!(\tilde{a}^{m+k})$, where \tilde{a} is the coupling class.

Theorem 1.4. [12] *Let (M, a) be a nilpotent connected c -symplectic manifold whose cohomology is isomorphic to $\mathbb{K}[a]/(a^{m+1})$. Then, as an algebra,*

$$H^*(B\text{aut}_1(M)) \cong \mathbb{K}[\mu_2, \dots, \mu_{m+1}],$$

where $\deg \mu_k = 2k$.

REFERENCES

- [1] A. K. Bousfield and V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, *Memoirs of AMS* **179**(1976).
- [2] E. H. Brown Jr and R. H. Szczarba, On the rational homotopy type of function spaces, *Trans. Amer. Math. Soc.* **349**(1997), 4931-4951.
- [3] U. Buijs and A. Murillo, Basic constructions in the rational homotopy theory of function spaces, *Annales de l'inst. Fourier* **56**(2006), 815-838.
- [4] U. Buijs and A. Murillo, The rational homotopy Lie algebra of function spaces, *Comment. Math. Helv.* **83**(2008), 723-739.
- [5] D. Chataur and K. Kuribayashi, An operadic model for a mapping space and its associated spectral sequence, *J. Pure and Appl. Algebra*, **210**(2007), 321-342.
- [6] D. H. Gottlieb, Evaluation subgroups of homotopy groups, *Amer. J. Math.* **91**(1969), 729-756.
- [7] A. Haefliger, Rational homotopy of space of sections of a nilpotent bundle, *Trans. Amer. Math. Soc.* **273**(1982), 609-620.
- [8] Y. Hirato, K. Kuribayashi and N. Oda, A function space model approach to the rational evaluation subgroups, *Math. Z.*, **258**(2008), 521-555.
- [9] T. Januszkiewicz and J. Kedra, Characteristic classes of smooth fibrations, preprint (2002) [arXiv:math/0209288v1](https://arxiv.org/abs/math/0209288v1).
- [10] J. Kedra and D. McDuff, Homotopy properties of Hamiltonian group actions, *Geometry & Topology*, **9**(2005), 121-162.
- [11] K. Kuribayashi, A rational model for the evaluation map, *Georgian Mathematical Journal*, **13**(2006), 127-141.
- [12] K. Kuribayashi, Rational visibility of a Lie group in the space of self-homotopy equivalences of a homogeneous space, preprint (2008).
- [13] G. Lupton and J. Oprea, Cohomologically symplectic space: Toral actions and the Gottlieb group, *Trans. Amer. Math. Soc.* **347**(1995), 261-288.
- [14] G. Lupton and S. B. Smith, Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G -sequences. *J. Pure Appl. Algebra* **209**(2007), 159-171.

- [15] G. Lupton and S. B. Smith, Rationalized evaluation subgroups of a map. II. Quillen models and adjoint maps, *J. Pure Appl. Algebra* **209**(2007), 173-188.
- [16] G. Lupton and S. B. Smith, The evaluation subgroup of a fibre inclusion, *Topology Appl.* **154**(2007), 1107-1118.
- [17] M. Mandell, E_∞ algebras and p -adic homotopy theory, *Topology* **40**(2001), no. 1, 43-94.
- [18] J. P. May, Classifying spaces and fibrations, *Mem. Amer. Math. Soc.* 155, 1975.
- [19] D. Quillen, Rational homotopy theory, *Ann. of Math.* **90**(1969), 205-295.
- [20] D. Sullivan, Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* No. 47 (1977), 269–331 (1978).
- [21] R. Thom, L'homologie des Espaces Fonctionnels. *Colloq. Top. Alg.*, Louvain 1956.
- [22] M. Vigué-Poirrier, Sur l'homotopie rationnelle des espaces fonctionnels, *Manuscripta Math.* **56**(1986) 177-191.

Maurer-Cartan moduli spaces and associated characteristic classes

ANDREY LAZAREV

(joint work with J. Chuang)

1. INTRODUCTION

1.1. Notation and conventions. We work in the category of $\mathbb{Z}/2$ -graded vector spaces (also known as super-vector spaces) over a field \mathbf{k} of characteristic zero. However all our results (with obvious modifications) continue to hold in the \mathbb{Z} -graded context. The adjective ‘differential graded’ will mean ‘differential $\mathbb{Z}/2$ -graded’ and will be abbreviated as ‘dg’. A (commutative) differential graded (Lie) algebra will be abbreviated as (c)dg(1)a. All of our unmarked tensors are understood to be taken over \mathbf{k} . For a $\mathbb{Z}/2$ -graded vector space $V = V_0 \oplus V_1$ the symbol ΠV will denote the *parity reversion* of V ; thus $(\Pi V)_0 = V_1$ while $(\Pi V)_1 = V_0$. The graded symmetric algebra on a dg vector space V is denoted by $S(V)$.

2. MAURER-CARTAN FUNCTOR AND CHEVALLEY-EILENBERG COMPLEXES

Definition 2.1. Let \mathfrak{g} be a dgla and A be a profinite cdga; then an element $\xi \in \mathfrak{g}^1 \otimes A$ is *Maurer-Cartan* if $d\xi + \frac{1}{2}[x, x] = 0$. The set of Maurer-Cartan elements in $\mathfrak{g}^1 \otimes A$ will be denoted by $\text{MC}(\mathfrak{g}, A)$. The correspondence $(\mathfrak{g}, A) \mapsto \text{MC}(\mathfrak{g}, A)$ is functorial in both variables.

If A is an arbitrary (not necessarily profinite) cdga then we will write $\widetilde{\text{MC}}(\mathfrak{g}, A)$ for the Maurer-Cartan functor. We will often suppress the tilde when the context makes the notation unambiguous.

Remark 2.2. We will consider $\text{MC}(\mathfrak{g}, A)$ as a functor in the second variable only although many statements will be symmetric in two variables; one can also consider associative version of the MC functor or even an MC functor based on a pair of Koszul dual operads.

Proposition 2.3.

- The functor $\text{MC}(\mathfrak{g}, A)$ is represented by the profinite cdga $\text{CE}^*(\mathfrak{g})$ (also known as the Chevalley-Eilenberg complex of \mathfrak{g}). The underlying vector space of $\text{CE}^*(\mathfrak{g})$ is $\hat{S}\Pi\mathfrak{g}$ whereas the differential d is the sum $d_{\text{int}} + d_{\text{CE}}$. Here d_{int} is the differential induced by the internal differential on d whereas the Chevalley-Eilenberg differential d_{CE} is a derivation whose restriction onto $\Pi\mathfrak{g}^*$ is a map $\Pi\mathfrak{g}^* \rightarrow \Pi\mathfrak{g}^* \otimes \Pi\mathfrak{g}^*$ induced by the commutator on \mathfrak{g} .
- The functor $\widetilde{\text{MC}}(\mathfrak{g}, A)$ is represented by the finite cdga $\widetilde{\text{CE}}^*(\mathfrak{g})$ (which we will refer to as the modified Chevalley-Eilenberg complex of \mathfrak{g}). The underlying vector space of $\widetilde{\text{CE}}(\mathfrak{g})$ is $S\Pi\mathfrak{g}^*$ whereas the differential is defined precisely as before.

Examples of Maurer-Cartan elements in algebra and geometry abound; a standard example is a flat connection on a vector bundle. Here is another, more recent example.

Let V be a dg vector space; consider the dgla $L(V) := \text{Der}_c(\hat{T}V^*)$ of continuous derivations of the completed tensor algebra on Π^* . A Maurer-Cartan element in $L(V)$ is the same as an A_∞ -algebra, [3], [1]. More precisely, an A_∞ -algebra is a derivation having no constant term, a more general notion is sometimes referred to as a *weak* A_∞ -algebra. An A_∞ -algebra is *cyclic* (or symplectic) if V is supplied with a (super) symplectic structure and ξ preserves it. The interest in cyclic A_∞ -algebras stems from Kontsevich's theorem [3] relating the stable cohomology of the Lie algebra of symplectic derivations to the homology of moduli spaces of Riemann surfaces.

Definition 2.4. Let $\xi \in \text{MC}(\mathfrak{g}, A)$; we have the corresponding classifying map of cdga's $\widetilde{\text{CE}}^*(\mathfrak{g}) \rightarrow A$. This gives rise to a map in cohomology $\widetilde{\text{HCE}}^*(\mathfrak{g}) \rightarrow H^*(A)$ or an element in $\widetilde{\text{HCE}}_*(\mathfrak{g}) \otimes H^*(A)$. This element is called the *characteristic class* of the Maurer-Cartan element ξ .

We will mostly consider the case $A = \mathbf{k}$, the ground field. It is known (cf. for example [4]) that characteristic classes of cyclic A_∞ -algebras could be nontrivial, e.g. they give rise to so-called tautological classes in the moduli spaces of Riemann surfaces. Moreover, classes of weakly equivalent A_∞ -algebras are the same, [1]. Our main result concerns with the characteristic classes of Morita equivalent A_∞ -algebras. Let m be a (cyclic) A_∞ -structure on a dg vector space V and $\text{End}(W)$ be the endomorphism algebra of some finite-dimensional dg space W . Then $V \otimes \text{End}(W)$ is itself a cyclic A_∞ -algebra whose characteristic classes are related in a very simple way to the characteristic class of (V, m) . Recall, [2] that any A_∞ -algebra which is Morita equivalent to (V, m) is obtained by twisting an A_∞ -structure on $V \otimes \text{End}(W)$. Then:

Theorem 2.5. *The characteristic class of a twisted A_∞ -structure on $V \otimes \text{End}(W)$ does not depend on the twisting (and thus, equals to that of the trivially twisted A_∞ -structure).*

This result shows that, suitably interpreted, the characteristic class of a cyclic A_∞ -algebra only depends on its dg-category of modules.

REFERENCES

- [1] A. Hamilton, A. Lazarev. *Characteristic classes of A_∞ -algebras*. J. Homotopy Relat. Struct. 3 (2008), no. 1, 65-111.
- [2] B. Keller, *Introduction to A-infinity Algebras and Modules*. Homology, Homotopy and Applications, Vol. 3, 2001, No. 1, pp. 1-35.
- [3] M. Kontsevich. *Formal noncommutative symplectic geometry*. The Gelfand Mathematical Seminars, 1990-1992, pp. 173–187, Birkhäuser Boston, Boston, MA, 1993.
- [4] G. Mondello, *Combinatorial classes on $\overline{\mathcal{M}}_{g,n}$ are tautological*. Int. Math. Res. Not. 2004, no. 44, 2329–2390.

Cyclic formality of the framed little discs operad

PAOLO SALVATORE

In this survey talk I first recalled the definition of operads and their algebras. In particular I recalled the connection of the little n -discs operad D_n to the theory of iterated loop spaces. The formality of the little D_n was proved by Kontsevich [2] (and Tamarkin [5] for $n = 2$). The result has applications to deformation quantization [2] and to homology computations for spaces of knots (Lambrechts, Turchin and Volic [3]). The little framed n -discs operad

fD_n (Getzler [1]) is also formal for $n = 2$. I proved this with Giansiracusa following Kontsevich's proof, and Severa did it following Tamarkin's proof [4]. Actually fD_2 is equivalent to a cyclic operad M .

This is the operad of moduli spaces of stable complex curves with tangent rays at the punctures and at the nodes. I proved with Giansiracusa the stronger result that M is formal in the category of cyclic operads.

REFERENCES

- [1] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories. Comm. Math. Phys. 159 (1994), no. 2, 265–285.
- [2] M. Kontsevich, Operads and motives in deformation quantization. Lett. Math. Phys. 48 (1999) 35–72.
- [3] P. Lambrechts, V. Turchin and I. Volic, The rational homology of spaces of long knots in codimension > 2 , arXiv:math/0703649
- [4] P. Severa, Formality of the chain operad of framed little disks, arXiv:math/0902.3576
- [5] D. Tamarkin, Formality of chain operad of little discs, Lett. Math. Phys. 66 (2003), n.1-2, 65-72

Miller spaces

JEFF STROM

The results of this talk appeared in the article [9].

In the seminal paper [8], Haynes Miller proved that the classifying spaces of locally finite groups G have a remarkable property: if K is any finite dimensional CW complex, then $\text{map}_*(BG, K)$ is weakly contractible. Our work begins when we define a space X to be a *Miller space* if for every finite *nilpotent* CW complex, $\text{map}_*(X, K) \sim *$, and ask: how can we recognize a Miller space?

The definition is easily modified to make sense in the stable category of spectra, and one easily sees that a spectrum X is a Miller spectrum if and only if the function spectra $F(X, S^n)$ are weakly contractible for all n (in fact, it is equivalent to have this for just one value of n). The purpose of the talk is to prove the following unstabilization of this elementary observation.

Theorem 1.1. *The following are equivalent:*

- (1) X is a Miller space
- (2) $\text{map}_*(X, S^n) \sim *$ for all sufficiently large n .

Clearly (1) implies (2), and the converse is where the work must be done.

The proof makes use of the formalism of strong resolving classes. A *strong resolving class* is a class \mathcal{R} of spaces which is closed under homotopy limits, weak homotopy equivalence and extensions by fibrations [9]. Resolving classes have a somewhat counterintuitive desuspension property: if $\bigvee_{i=1}^m \Sigma^N K \in \mathcal{R}$ for some N and all m , then $K \in \mathcal{R}$. [2, 7, 3]

For our purposes, the main example of a strong resolving class is, for a fixed space X , the class

$$\mathcal{R} = \{K \mid \text{map}_*(X, K) \sim *\}.$$

Strong resolving classes of this form have an additional closure property: they are closed under the formation of wedges $\bigvee K_\alpha$, provided that for each n only finitely many of the K_α are not n -connected (such wedges are called *finite type wedges*) [4].

Together with the desuspension property enjoyed by all resolving classes, this implies that if \mathcal{A} is a collection of spaces with $\Sigma^N \mathcal{A} = \{\Sigma^N A \mid A \in \mathcal{A}\} \subseteq \mathcal{R}$, then $\mathcal{A} \subseteq \mathcal{R}$. Taking \mathcal{A} to be the collection of all spheres, we see that if X satisfies condition (2) of the theorem, then $S^n \in \mathcal{R}$ for all n . Even more, all finite type wedges of spheres are contained in \mathcal{R} .

With these preliminaries in place, the proof proceeds by induction on the cone length of K with respect to finite type wedges of spheres [1]. If K has cone length 1, then it is a finite type wedge of spheres, and so $\text{map}_*(X, K) \sim *$. For the inductive step, we find a cofibration sequence

$$L \rightarrow K \rightarrow \bigvee S^{n_\alpha},$$

where the cone length of L is strictly less than that of K . Next we convert the map $K \rightarrow \bigvee S^{n_\alpha}$ to a fibration and form the homotopy fiber F . Since \mathcal{R} is closed under extensions by fibrations and $\bigvee S^{n_\alpha} \in \mathcal{R}$, it suffices to prove that $F \in \mathcal{R}$.

But it follows from a result of Ganea (see also Gray) [5, 6] that the spherical cone length of ΣF is bounded above by that of L ; and so the same is true for $\bigvee_{i=1}^m \Sigma F$ for all m . Now the desuspension property of resolving classes yields $F \in \mathcal{R}$ and hence $K \in \mathcal{R}$, completing the inductive step.

REFERENCES

- [1] M. Arkowitz, D. Stanley, J. Strom, *The \mathcal{A} -category and \mathcal{A} -cone length of a map*, in: Lusternik–Schnirelmann Category and Related Topics (South Hadley, MA, 2001), Contemp. Math. 316, Amer. Math. Soc., Providence, RI, 2002, 15–33.
- [2] M. G. Barratt, *The spectral sequence of an inclusion*, in: Colloquium in Algebraic Topology, Matematisk Institut, Aarhus Universitet, 1962, 22–27.
- [3] A. K. Bousfield, *On the homology spectral sequence of a cosimplicial space*. Amer. J. Math. **109** (1987), no. 2, 361–394.
- [4] W. Dwyer, Private communication.
- [5] T. Ganea, *A generalization of the homology and homotopy suspension*. Comment. Math. Helv. **39** (1965) 295–322.
- [6] B. Gray, *On the homotopy groups of mapping cones*. Proc. London Math. Soc. (3) **26** (1973), 497–520.
- [7] M. J. Hopkins, *Formulations of cocategory and the iterated suspension*, Algebraic homotopy and local algebra (Luminy, 1982), 212–226, Astrisque, **113-114**, Soc. Math. France, Paris, 1984.
- [8] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. **120** (1984), 39–87.
- [9] J. Strom, *Miller spaces and spherical resolvability of finite complexes*. Fund. Math. **178** (2003), no. 2, 97–108.

Integral loop homology of complete flag manifolds

SVJETLANA TERZIĆ

(joint work with Jelena Grbić)

I presented the recent results (see [1]) related to explicit description of the integral Pontrjagin homology of the based loop space on a complete flag manifold G/T . In my talk I explicitly formulated this result in the case when $G = SU(n+1)$. The similar statements are also proved in [1] for the other classical Lie groups and for the exceptional groups G_2 , F_4 and E_6 .

1. RATIONAL LOOP SPACE HOMOLOGY OF COMPLETE FLAG MANIFOLDS

The rational homology ring structure of these spaces can be described by making use of Sullivan’s minimal model theory and Milnor-Moore theorem, see [2]. Denote by L_X the rational homotopy Lie algebra of the simply connected topological space X of finite type. Milnor and Moore theorem states that the algebra $H_*(\Omega(X); \mathbb{Q})$ is isomorphic to the universal enveloping algebra of L_X . On the other hand if $\mu = (\Lambda V, d)$ denotes Sullivan minimal model for X , then L_X is, as a Lie algebra, isomorphic to the homotopy Lie algebra \mathcal{L} of μ . The algebra \mathcal{L} is defined as follows. The underlying vector space L is given with $sL = \text{Hom}(V; \mathbb{Q})$, where s is the usual suspension. In order to introduce the Lie brackets let us denote by $d_1 : V \rightarrow \Lambda^2 V$ the quadratic part of the differential d . One can define a pairing $\langle ; \rangle : V \times sL \rightarrow \mathbb{Q}$ by $\langle v; sx \rangle = (-1)^{\deg v} sx(v)$ and extend it to $\Lambda^k V \times sL \times \cdots \times sL \rightarrow \mathbb{Q}$ by letting $\langle v_1 \wedge$

$\cdots \wedge v_k; sx_k, \dots, sx_1\rangle = \sum_{\sigma \in S_k} \epsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \cdots \langle v_{\sigma(k)}; sx_k\rangle$, where S_k is the symmetric group and $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \epsilon_\sigma v_1 \wedge \cdots \wedge v_k$. Then L inherits a Lie brackets from d_1 uniquely determined by $\langle v; s[x, y]\rangle = (-1)^{\deg y + 1} \langle d_1 v; sx, sy\rangle$ for $x, y \in L, v \in V$. It follows that $H_*(\Omega X; \mathbb{Q}) \cong U\mathcal{L}$ where $U\mathcal{L}$ is the universal enveloping algebra for \mathcal{L} . Further on, $U\mathcal{L} \cong T(L)/\langle xy - (-1)^{\deg x \deg y} yx - [x, y]\rangle$.

When $X = G/T$, where G is a compact connected Lie groups and T its maximal torus, we have that X is formal and its rational cohomology algebra is given with $H^*(G/T; \mathbb{Q}) \cong \mathbb{Q}[u_1, \dots, u_n]/\langle \mathbb{Q}[u_1, \dots, u_n]^{W_G} \rangle$. If G is a simple Lie group the Weyl invariant polynomials that generate $\mathbb{Q}[u_1, \dots, u_n]^{W_G}$ are established and, therefore, the described procedure leads to the explicit computation of the rational homology of $\Omega(G/T)$.

We provide this computation for $X = SU(n+1)/T^n$. It is known that the rational (as well as integral) cohomology of $SU(n+1)/T^n$ is

$$H^*(SU(n+1)/T^n; \mathbb{Q}) \cong \mathbb{Q}[u_1, \dots, u_{n+1}]/\langle S^+(u_1, \dots, u_{n+1}) \rangle,$$

where $\deg u_i = 2, 1 \leq i \leq n+1$. As the ideal $\langle S^+(u_1, \dots, u_{n+1}) \rangle$ is a Borel ideal, the minimal model for $SU(n+1)/T^n$ is by [4] given with $\mu = (\Lambda V, d)$, where $V = (u_1, \dots, u_n, v_1, \dots, v_n)$ and $\deg(u_k) = 2, \deg(v_k) = 2k+1$ for $1 \leq k \leq n$. The differential d is defined by $d(u_k) = 0, d(v_k) = \sum_{i=1}^n u_i^{k+1} + (-1)^{k+1} \left(\sum_{i=1}^n u_i \right)^{k+1}$, what implies that the its quadratic part is only non trivial on v_2 , i.e. $d_1(v_1) = 2 \sum_{i=1}^n u_i^2 + 2 \sum_{i < j} u_i u_j$. If we now apply the given procedure we obtain the following.

Theorem 1.1. *The rational homology ring of the loop space on the flag manifold $SU(n+1)/T^n$ is*

$$(1) \quad H_*(\Omega(SU(n+1)/T^n); \mathbb{Q}) \cong (T(a_1, \dots, a_n)/\langle a_k^2 = a_p a_q + a_q a_p \mid 1 \leq k, p, q \leq n, p \neq q \rangle) \otimes \mathbb{Q}[b_2, \dots, b_n]$$

where the generators a_i are of degree 1 for $1 \leq i \leq n$, and the generators b_k are of degree $2k$ for $2 \leq k \leq n$.

2. INTEGRAL LOOP SPACE HOMOLOGY OF COMPLETE FLAG MANIFOLDS

We describe the integral Pontrjagin homology ring structure on $\Omega(G/T)$ where G is a compact simple Lie group by showing that there is a split extension of algebras

$$1 \longrightarrow H_*(\Omega_0 G; \mathbb{Z}) \longrightarrow H_*(\Omega(G/T); \mathbb{Z}) \longrightarrow H_*(T; \mathbb{Z}) \longrightarrow 1.$$

In order to show this extension we first prove the following [1].

Theorem 2.1. *The homology of the based loop space on the complete flag manifold of a compact connected Lie group is torsion free.*

We decide then the extension as follows, by making use of the rational homology calculations we have done for $\Omega(G/T)$, and the results on integral homology of the identity component $\Omega_0 G$ of the loop space on G , which is known to be torsion free.

Note that for any simply connected Lie group G we have that $\pi_2(G/T) \cong \mathbb{Z}^{\dim T}$ and $\pi_3(G/T) \cong \mathbb{Z}$. If we identify $H_1(T, \mathbb{Z})$ with $\pi_2(G/T)$ and $H_2(\Omega G, \mathbb{Z})$ with $\pi_3(G/T)$ via natural homomorphisms we obtain the extension in the above splitting to be given with $[\alpha, \beta] = W(\alpha, \beta) \in H_2(\Omega SU(n+1); \mathbb{Z})$, where $\alpha, \beta \in H_1(T^n; \mathbb{Z})$ and $W: \pi_2(G/T) \otimes \pi_2(G/T) \rightarrow \pi_3(G/T)$ is the pairing given by the Whitehead product.

In the case when $G = SU(n+1)$ it is known [3] that the subspace of primitive elements in $H_*(\Omega SU(n+1); \mathbb{Z})$ is spanned by the elements $\sigma_1, \dots, \sigma_n$ which can be expressed in terms of integral generators y_1, \dots, y_n of $H_*(\Omega SU(n+1); \mathbb{Z})$ using the Newton formula

$$(2) \quad \sigma_k = \sum_{i=1}^{k-1} (-1)^{i-1} \sigma_{k-i} y_i + (-1)^{k-1} k y_k, \quad 1 \leq k \leq n.$$

These primitive elements $\sigma_1, \dots, \sigma_n$ rationalise to b_1, \dots, b_n from rational homology calculations.

In this way we obtain that the above splitting determines the following integral Pontrjagin homology of $\Omega(SU(n+1)/T^n)$.

Theorem 2.2. *The integral Pontrjagin homology ring of the loop space on $SU(n+1)/T^n$ is*

$$H_*(\Omega(SU(n+1)/T^n); \mathbb{Z}) \cong (T(x_1, \dots, x_n) \otimes \mathbb{Z}[y_1, \dots, y_n]) / \langle x_k^2 = x_p x_q + x_q x_p = 2y_1 \text{ for } 1 \leq k, p, q \leq n, p \neq q \rangle$$

where the generators x_1, \dots, x_n are of degree 1, and the generators y_i are of degree $2i$ for $1 \leq i \leq n$.

REFERENCES

- [1] J. Grbić and S. Terzić, *The integral Pontrjagin homology of the based loop space on a flag manifold*, Osaka Journal of Mathematics, to appear; arXiv:math.AT/0702113.
- [2] Y. Félix, S. Halperin and J. C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics **205**, Springer-Verlag, 2001.
- [3] R. Bott, *The space of loops on a Lie group*, Michigan Mathematical Journal, **5** (1958), 35–61.
- [4] A. Bousfield and V. Gugenheim, *On PL de Rham theory and rational homotopy type*, Memoirs of the American Mathematical Society **179** (1976), ix–94.

On the configuration spaces of a certain n -arms machine in the Euclidean space

SHUICHI TSUKUDA

(joint work with Yasuhiko Kamiyama)

We describe the homotopy type of the configuration space of a certain arachnoid mechanism – that is, a parallel robot in \mathbb{R}^d having n two-joined legs, with all joints of a fixed length $a/2$, joined together at a central point q , with the other end of the i -th leg at the i -th vertex of a regular polyhedron P .

More precisely, we consider the configuration space $\mathcal{M}(P, a)$ defined as follows: Let $P \subset \mathbb{R}^3 \subset \mathbb{R}^d$ be a regular convex polyhedron with vertices $\{v_1, \dots, v_n\}$. We set

$$\mathcal{M}(P, a) := \left\{ (p_1, \dots, p_n, q) \mid \|p_i - v_i\| = \|q - p_i\| = \frac{a}{2}, \forall i \right\} \subset (\mathbb{R}^d)^{n+1}$$

We say that the machine has short arms if $l(P) < a < L(P)$ and long ones if $a > L(P)$, where $l(P)$ and $L(P)$ are the radius and the diameter of P , respectively.

Theorem 1.1. *If the arms are short and $d \geq 4$, then*

$$\mathcal{M}(P, a) \simeq \bigvee_{\emptyset \neq I \subset V} S^{|I|(d-2)+d-3} \wedge S(I^\Delta)$$

where V is the set of vertices of P , I^Δ denotes the union of the faces of the dual polyhedron P^Δ those correspond to vertices in I and $S(I^\Delta)$ stands for the unreduced suspension of I^Δ (with the base point $t = 0$). Moreover, the right hand side is a bouquet of spheres and we can explicitly determine the numbers and the dimensions of them. If $d = 3$, the same decomposition holds after a single suspension.

When $d = 3$, we can determine the integral homology groups of $\mathcal{M}(P, a)$ for the long arms machines. The proof depends on the description of the moduli space as a certain homotopy colimit, which is closely related to the recent work of [1].

REFERENCES

- [1] A. Bahri, M. Bendersky, F. R. Cohen, S. Gitler, *The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces*, arXiv:0711.4689
- [2] Y. Kamiyama and S. Tsukuda, *The configuration space of the n -arms machine in the Euclidean space*, *Topology Appl.* **154** (2007), 1447–1464.
- [3] Y. Kamiyama and S. Tsukuda, *On the homology of configuration spaces of arachnoid mechanisms*, *Houston J. Math.* **34** (2008), 483–499.

Equivalences of a product and Mal'cev quasirings

ANTONIO VIRUEL

Let X be a pointed space and let $\mathcal{E}(X)$ denote the group of based homotopy classes of based homotopy equivalences of X into itself. The study of $\mathcal{E}(X \times Y)$ is one of the major problems for people studying self-homotopy equivalences, and it was attacked by P. Pavešić [3] by introducing two distinguished subsets:

- $\mathcal{E}_X(X \times Y) \subset \mathcal{E}(X \times Y)$ is the set of all homotopy equivalences of the form (p_X, f) , where $p_X : X \times Y \rightarrow X$ is the projection and $f : X \times Y \rightarrow Y$ is a continuous map.
- Similarly $\mathcal{E}_Y(X \times Y) \subset \mathcal{E}(X \times Y)$ is defined as all the homotopy equivalences of the form (g, p_Y) .

In the list of problems on self-homotopy equivalences compiled by Arkowitz [1], Pavešić asks if $\mathcal{E}(X \times Y)$ is generated by $\mathcal{E}_X(X \times Y)$ and $\mathcal{E}_Y(X \times Y)$ (Problem 13th.). The aim of this work is to show that $\mathcal{E}(X \times Y)$ is not generated by $\mathcal{E}_X(X \times Y)$ and $\mathcal{E}_Y(X \times Y)$ in general. We consider the case when X equals Y , and in order to avoid confusion, we shall denote $\mathcal{E}_1(X^2) = \mathcal{E}_X(X \times Y)$ and $\mathcal{E}_2(X^2) = \mathcal{E}_Y(X \times Y)$ for $X = Y$. Then we prove:

Theorem 1.1. *Let X be a space for which $\mathcal{E}(X^2)$ is generated by $\mathcal{E}_1(X^2)$ and $\mathcal{E}_2(X^2)$, then any Whitehead bracket in π_*X vanishes.*

As there exist spaces X such that the Whitehead brackets in π_*X don't vanish (as those whose fundamental group is not abelian), $\mathcal{E}(X^2)$ is not generated by $\mathcal{E}_1(X^2)$ and $\mathcal{E}_2(X^2)$ in general. This answers P. Pavešić's question in a negative way.

In [2], Mal'cev introduced the concept of quasi-ring as a unifying framework in which results on groups and rings could be proved simultaneously.

In order to prove Theorem 1.1, we consider the similar problem in the category of quasirings and then we apply the functor that send a space to the quasiring given by its homotopy groups with Whitehead brackets.

REFERENCES

- [1] M. Arkowitz, *Problems on Self-Homotopy Equivalences*, Contemp. Math. **274** (2001), 309–315.
- [2] A.I. Mal'cev, *On a class of algebraic systems*. Uspehi Matem. Nauk (N.S.) **8** (1953), no. 1(53), 165–171.
- [3] P. Pavešić, *Self fibre homotopy equivalences of fibred product*, Topology Appl. **102** (2000), 169–180.

Non-integral central extensions of loop groups via gerbes

CHRISTOPH WOCKEL

Smooth loop groups $\Omega^\infty G := C_*^\infty(S^1, G)$ for G a compact, simple and simply connected Lie group, have intensively been studied as infinite-dimensional Lie groups. However, in many situations (e.g., in order to develop a reasonable representation theory), it is convenient not to consider $\Omega^\infty G$, but its universal central extension

$$1 \rightarrow \mathbb{T} \rightarrow \widehat{\Omega^\infty G} \rightarrow \Omega^\infty G$$

for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the one-dimensional torus group. On the Lie algebra $L(\Omega^\infty G) = \Omega^\infty \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G , this central extension is given by the Kac-Moody cocycle

$$\omega : \Omega^\infty \mathfrak{g} \times \Omega^\infty \mathfrak{g} \rightarrow \mathbb{R}, \quad (f, g) \mapsto \int_0^1 \langle f, g' \rangle dt,$$

where $\langle \cdot, \cdot \rangle \in S^2 \mathfrak{g}^*$ is chosen such that the left-invariant extension of $\langle \cdot, [\cdot, \cdot] \rangle$ generates $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$. Then ω gives rise to the central extension

$$\mathbb{R} \rightarrow \mathbb{R} \oplus_\omega \Omega^\infty \mathfrak{g} \rightarrow \Omega^\infty \mathfrak{g},$$

where the Lie algebra structure on $\mathbb{R} \oplus_{\omega} \Omega^{\infty} \mathfrak{g}$ is given by

$$[(x, f), (y, g)] := (\omega(f, g), [f, g])$$

(see [PS86], [Nee02] for generalities on central extensions of Lie groups and algebras, [MN03] for generalisations to arbitrary mapping groups and algebras and [NW09] for generalisations to gauge groups and groups of sections in Lie group bundles).

The question is for which values of $t \in \mathbb{R}$ the central extension $\mathbb{R} \oplus_{t \cdot \omega} \Omega^{\infty} \mathfrak{g}$ of Lie algebras actually comes from a central extension $U(1) \rightarrow \widehat{\Omega^{\infty} G}_t \rightarrow \Omega^{\infty} G$ of Lie groups. This was solved by Pressley and Segal in [PS86, Th. 4.4.1], see also [Mic87], [Mur88], [MS01], [Nee02, Prop. 5.11, Th. 7.9] [MN03, Th. I.6, Th. II.9] and [NW09, Th. 2.12, Th. 3.14]:

Theorem. The central extension $\widehat{\Omega^{\infty} G}_t$ exists if and only if $t \in \mathbb{Z}$. In this case, the connecting homomorphism

$$\mathbb{Z} \cong \pi_3(G) \cong \pi_2(\Omega G) \cong \pi_2(\Omega^{\infty} G) \xrightarrow{\delta_2} \pi_1(U(1)) \cong \mathbb{Z}$$

of the fibration $U(1) \rightarrow \widehat{\Omega^{\infty} G}_t \rightarrow \Omega^{\infty} G$ is given by $z \mapsto tz$. In particular, we have $\pi_2(\widehat{\Omega^{\infty} G}_t) \cong \mathbb{Z}/t\mathbb{Z}$.

Thus $\widehat{\Omega^{\infty} G}_1$ can be viewed as a 2-connected cover of $\Omega^{\infty} G$, where cover has to be interpreted appropriately for we do not extend by a discrete group.

The question what happens for non-integral values of t does not seem to be appropriate from this point view for the conditions on and implications from $\langle \cdot, \cdot \rangle$ and ω seem to dictate integrality. However, by passing from central extensions of Lie groups to abelian multiplicative principal 2-bundles and rephrasing the results from [Woc08b] we can show the following:

Theorem For each $t \in \mathbb{R}$ there exists a multiplicative principal 2-bundle $\widehat{\mathcal{G}}_t$ over $\Omega^{\infty} G$, together with a flat (but non-fake-flat) connection such that the induced 2-holonomy is given by the period homomorphism

$$\text{per}_{t\omega} : \pi_2(\Omega^{\infty} G) \rightarrow \mathbb{R}$$

(cf. [Nee02, Sect. 5] for the construction of $\text{per}_{t\omega}$, [Woc08a] for principal 2-bundles and their connection to gerbes, [Wal08] and [MS03] for multiplicative gerbes and [SW08] and [BM05] for connections on principal 2-bundles and gerbes).

REFERENCES

- [BM05] Lawrence Breen and William Messing. Differential geometry of gerbes. *Adv. Math.*, 198(2):732–846, 2005.
- [Mic87] Jouko Mickelsson. Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization. *Comm. Math. Phys.*, 110(2):173–183, 1987.
- [MN03] Peter Maier and Karl-Hermann Neeb. Central extensions of current groups. *Math. Ann.*, 326(2):367–415, 2003.
- [MS01] Michael K. Murray and Daniel Stevenson. Yet another construction of the central extension of the loop group. In *Geometric analysis and applications (Canberra, 2000)*,

- volume 39 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 194–200. Austral. Nat. Univ., Canberra, 2001.
- [MS03] Michael K. Murray and Daniel Stevenson. Higgs fields, bundle gerbes and string structures. *Comm. Math. Phys.*, 243(3):541–555, 2003.
- [Mur88] Michael K. Murray. Another construction of the central extension of the loop group. *Comm. Math. Phys.*, 116(1):73–80, 1988.
- [Nee02] Karl-Hermann Neeb. Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)*, 52(5):1365–1442, 2002.
- [NW09] Karl-Hermann Neeb and Christoph Wockel. Central extensions of groups of sections, 2009, arXiv:0711.3437. To appear in *Ann. Glob. Anal. Geom.*
- [PS86] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.
- [SW08] Urs Schreiber and Konrad Waldorf. Connections on non-abelian Gerbes and their Holonomy, 2008, arXiv:0808.1923.
- [Wal08] Konrad Waldorf. Multiplicative Bundle Gerbes with Connection, 2008, arXiv:0804.4835.
- [Woc08a] Christoph Wockel. A global perspective to gerbes and their gauge stacks, 2008, arXiv:0803.3692.
- [Woc08b] Christoph Wockel. Categorical central extensions, étale Lie 2-groups and Lie’s Third Theorem for locally exponential Lie algebras, 2008, arXiv:0812.1673.