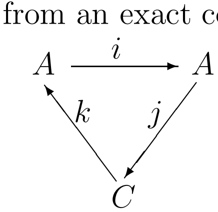
A Based Federer Spectral Sequence and the Rational Homotopy of Function Spaces

Samuel B. Smith
Department of Mathematics
Saint Joseph's University
Philadelphia, PA 19131

We study the rational homotopy of function spaces within the context of Quillen's minimal models. Our method is to consider a spectral sequence with $E_2^{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to the rational homotopy groups of components of the based function space $M(X,Y)_*$. Our results include calculations of rational homotopy groups as well as general contributions to the rational classification problem for components of function spaces.

1. An Exact Couple. Let $M(X,Y)$ and $M(X,Y)_*$ denote, respectively, the spaces of free and based continuous functions between two spaces X and Y. As an overriding assumption, all spaces are taken to be simply connected CW complexes. We denote the path components corresponding to a map $f : X \to Y$ by $M_f(X,Y)$ and $M_f(X,Y)_*$.

The based Federer spectral sequence for a map $f : X \to Y$ arises, like the original [1], from an exact couple of the form



Suppose X comes equipped with a fixed CW decomposition so that for each $q > 1$ there is a cofibration sequence

$$\bigvee_\alpha S^{q-1}_\alpha \xrightarrow{h_q} X^{q-1} \to X^q,$$

where X^q is the q-skeleton of X and h_q is the wedge of the attaching maps for the q-cells of X. Let $f_q : X^q \to Y$ denote the restriction of f and let $W^{q-1} = \bigvee_\alpha S^{q-1}_\alpha$. We obtain a long exact sequence on homotopy of which a portion is

$$\pi_{p+1}(M_0(W^{q-1}, Y)_*) \xrightarrow{\partial_q} \pi_p(M_{f_q}(X^q, Y)_*) \xrightarrow{(\rho_q)_*}$$

$$\pi_p(M_{f_{q-1}}(X^{q-1}, Y)_*) \xrightarrow{(\overline{h_q})_*} \pi_p(M_0(W^{q-1}, Y)_*).$$

Set $A_{p,q} = \pi_p(M_{f_q}(X^q, Y)_*)$, $C_{p,q} = \pi_{p+1}(M_0(W^{q-1}, Y)_*)$, and let $i = (\rho_q)_*$, $j = (\overline{h_q})_*$ and $k = \partial_q$. By adjointness, $C_{p,q} \cong \bigoplus_\alpha \pi_{p+1}(\Omega^{q-1}(Y)) \cong \widetilde{C}^q(X, \pi_{p+q}(Y))$, the reduced cellular q-cochains of X. When X is a finite complex, by [6, Theorem 2.3] we may replace X by any rationally equivalent space without affecting the rational homotopy of $M_f(X, Y)_*$. In particular, we may assume X comes equipped with a minimal CW decomposition with respect to its rational homotopy type. In this case, for degree reasons there are no nontrivial coboundary relations in the cellular cochain complex for X and so $C_{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y))$. Thus we have

Theorem 1.1 *Let X be finite and $f : X \to Y$ a based map. Then there is a spectral sequence with $E^2_{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to $\pi_p(M_f(X, Y)_*) \otimes \mathbb{Q}$.* $\square$

2. Null Components. We prove that the spectral sequence collapses on null components. The key lemma here is

Lemma 2.1 *Let X have dimension n and suppose $\alpha \in \pi_m(X)$ for $m > n$. Then $\Sigma(\alpha)$ is of finite order in $\pi_{m+1}(\Sigma(X))$.*

Proof. Let $f : S^m \to X$ represent α and let T_f denote the mapping cone of f. Since $\Sigma(T_f) \simeq T_{\Sigma f}$ it follows that $\Sigma(T_f)$ admits a minimal CW decomposition for which Σf is the attaching map for the top cell. But $\Sigma(T_f)$ is rationally equivalent to a wedge of spheres and so the Quillen minimal model for $\Sigma(T_f)$ has trivial differential. By [3, Proposition 8.12] the differential in the Quillen model of $\Sigma(T_f)$ is determined by the attaching maps in a minimal CW decomposition. Thus Σf is rationally trivial. $\square$

Theorem 2.2 *Let X be a finite complex and Y any space. Then*

$$\pi_q(M_0(X,Y)_*) \otimes \mathbb{Q} \cong \bigoplus_{k=2}^{\infty} \widetilde{H}^{k-q}(X, \pi_k(Y) \otimes \mathbb{Q}) \quad \text{and}$$

$$\pi_q(M_0(X,Y)) \otimes \mathbb{Q} \cong \bigoplus_{k=2}^{\infty} H^{k-q}(X, \pi_k(Y) \otimes \mathbb{Q}).$$

Proof. Since $\pi_*(M_0(X,Y)) \cong \pi_*(M_0(X,Y)_*) \oplus \pi_*(Y)$, the second line is a consequence of the first. For the first line, we note that when f is trivial, the map $j = (\overline{h_q})_* : A_{p,q-1} \to C_{p-1,q}$ is given by $j(\beta)(\sigma_i) = \beta \circ \Sigma^p(\alpha_i)$, for $\beta \in A_{p,q-1} \cong [\Sigma^p(X^{q-1}), Y]$, where here the σ_i denote the q-cells of X and the $\alpha_i \in \pi_{q-1}(X^{q-1})$ denote the corresponding homotopy elements. Thus the rational spectral sequence collapses by Lemma 2.1. $\square$

The original Federer spectral sequence rarely collapses for nontrivial components. In fact we have

Theorem 2.3 *Given $f : X \to Y$ with either X finite or Y rationally finite-dimensional, if there exists $\alpha \in \pi_n(X) \otimes \mathbb{Q}$ and $\beta \in \pi_m(Y) \otimes \mathbb{Q}$ with $[f_*(\alpha), \beta]_w \neq 0$ in $\pi_{n+m-1}(Y) \otimes \mathbb{Q}$ then $M_f(X,Y) \not\simeq_{\mathbb{Q}} M_0(X,Y)$.*

Proof. By Theorem 2.2 (or [5, Lemma 3.2] when Y is finite-dimensional) it suffices to show that $\beta \notin \rho_*(\pi_m(M_f(X,Y)) \otimes \mathbb{Q})$ where $\rho : M_f(X,Y) \to Y$ is the evaluation fibration. Let Y_0 denote the rationalization of Y. Then $M_f(X,Y)$ is (weakly) rationally equivalent to $M_f(X,Y_0)$. If $\beta \in \rho_*(\pi_m(M_f(X,Y)) \otimes \mathbb{Q})$ then there exists a map $F : S^m \times X \to Y_0$ with $F(s,*) = g(s)$ and $F(*,x) = f(x)$ where $g : S^m \to Y_0$ represents β. Let $h : S^n \to X$ represent α and define $H : S^m \times S^n \to Y_0$ by $H(s,t) = F(s,h(t))$. Note that H extends $g \vee (f \circ h) : S^m \bigvee S^n \to Y_0$. Thus $[f_*(\alpha), \beta]_w = 0$ in $\pi_{n+m-1}(Y_0)$ contradicting our hypothesis. $\square$

3. The First Differentials. Suppose X has a minimal rational CW decomposition with first nontrivial attaching maps in degree $m+1$. Let $W^m \xrightarrow{h_m} X^m \longrightarrow X^{m+1}$, be the corresponding cofibration sequence with $W^m = \bigvee_{i=1}^s S^m$ and $X^m = \bigvee_{j=1}^t S^{n_j}$ where each $n_j \leq m$. The first potentially nonzero differentials for $f : X \to Y$ are then the $d_{(m-n_j+1)} : C_{p,n_j} \to C_{p-1,m+1}$ for $j = 1, \ldots, t$. Let $\alpha_1, \ldots, \alpha_s \in \pi_m(X^m)$ be the homotopy classes represented by h_m and let $a_1, \ldots, a_s \in \pi_m(X^m) \otimes \mathbb{Q}$ be the corresponding rational homotopy classes. Let $x \in \pi_p(S^p) \otimes \mathbb{Q}$ and $y_j \in \pi_{n_j}(S^{n_j}) \otimes \mathbb{Q}$ for $j = 1, \ldots, t$ be nontrivial elements. By Hilton's Theorem, each a_i is decomposable into Whitehead products involving the y_j. Use the Jacobi identity in the Whitehead algebra $\pi_*(S^p \bigvee X^m) \otimes \mathbb{Q}$ to write

$$[x, a_i]_w = \sum_{j=1}^{t} [[x, y_j]_w, g_{ij}]_w, \tag{1}$$

for each i and some $g_{ij} \in \pi_{m-n_j+1}(X^m) \otimes \mathbb{Q}$. Recalling that $C_{p,n_j} \cong \widetilde{C}^{n_j}(X, \pi_{p+n_j}(Y_0))$, given $b_j \in \pi_{p+n_j}(Y_0)$ write b_j^* for the cochain which takes the value b_j on the cell corresponding to S^{n_j} and vanishes on the other cells of X. With this notation we prove

Theorem 3.1 *The differentials*

$$d_{(m-n_j+1)} : \widetilde{C}^{n_j}(X, \pi_{p+n_j}(Y_0)) \to \widetilde{C}^{m+1}(X, \pi_{p+m}(Y_0))$$

are given, for $j = 1, \ldots, t$, by

$$d_{(m-n_j+1)}(b_j^*)(\sigma_i) = [b_j, f_*(g_{ij})]_w,$$

where $\sigma_1, \ldots, \sigma_s$ are the $m+1$-cells of X.

Proof. If $a \in \pi_n(X) \otimes \mathbb{Q}$ then $\overline{a}$ will denote a representative in the Quillen model (L_X, ∂_X) of X for the corresponding element of $\pi_{n-1}(\Omega X) \otimes \mathbb{Q}$. If V is a graded vector space then $F(V)$ will denote the free graded Lie algebra generated by V. The differential $d_{(m-n_j+1)} = (\overline{h_m})_* \circ ((\rho_m)_* \circ \cdots \circ (\rho_{n_j+1})_*)^{-1} \circ \partial_{n_j}$. The proof consists in unravelling this composition.

Write the Quillen model for $S^p \times X_m$ as

$$(L_{S^p\times X_m}, \partial_{S^p\times X_m}) = (F(\mathbb{Q}(\overline{x}) \oplus V_{(m)} \oplus s^p(V_{(m)})), \partial_{S^p\times X_m})$$

where $V_{(m)}$ is the graded vector space generated by the elements $\overline{y}_j$ and $s^p(V_{(m)})$ is the pth suspension of $V_{(m)}$. The differential $\partial_{S^p\times X_m}$ vanishes on $\overline{x}$ and $V_{(m)}$ and is given on $s^p(V_{(m)})$ as $\partial_{S^p\times X_m}(s^p(\overline{y}_j)) = [\overline{x}, \overline{y}_j]$. The element $((\rho_m)_* \circ \cdots \circ (\rho_{n_j+1})_*)^{-1} \circ \partial_{n_j}(b_j^*) \in A_{p,m}$ is represented by the map $F_m : S^p \times X^m \to Y_0$ given on Quillen models by $(F_m)_*(\overline{x}) = 0, (F_m)_*(\overline{y}_i) = (f_m)_*(\overline{y}_i)$ and $(F_m)_*(s^p(\overline{y}_i)) = \delta_{ij} \cdot \overline{b}_j$.

Note that $F_m \circ (1 \times h_m) : S^p \times W^m \to Y_0$ is null on the subspace $S^p \bigvee W^m$ and so induces $G : \Sigma^p(W^m) \to Y_0$. The element $d_{m-n_j+1}(b_j^*)(\sigma_i) \in \pi_{p+m}(Y_0)$ is represented by the restriction of G to the ith sphere of the wedge $\Sigma^p(W^m) = \bigvee_{k=1}^s S^{m+p}$. Write the Quillen model for $S^p \times W^m$ as

$$(L_{S^p\times W^m}, \partial_{S^p\times W^m}) = (F(\mathbb{Q}(\overline{x}) \oplus V_{m-1} \oplus s^p(V_{m-1})), \partial_{S^p\times W^m})$$

where here V_{m-1} is concentrated in degree $m-1$ and generated by elements $\overline{u}_k$ for $k = 1, \ldots, s$. The Quillen model for $\Sigma^p(W^m)$ is just $(F(s^p(V_{m-1})), 0)$, the graded Lie algebra with trivial boundary. Thus $(F_m)_* \circ (1 \times (h_m)_*)(s^p(\overline{u}_i)) \in (L_Y, \partial_Y)$ is a Quillen model representative for $d_{m-n_j+1}(b_j^*)(\sigma_i)$.

Now $1 \times (h_m)_* : (L_{S^p\times W^m}, \partial_{S^p\times W^m}) \to (L_{S^p\times X_m}, \partial_{S^p\times X_m})$ satisfies $(h_m)_*(\overline{u}_i) = \overline{a}_i$ and so

$$\begin{aligned}\partial_{S^p\times X^m}(1 \times (h_m)_*)(s^p(\overline{u}_i)) &= (1 \times (h_m)_*)(\partial_{S^p\times W^m}(s^p(\overline{u}_i)) \\ &= (1 \times (h_m)_*)([\overline{x}, \overline{u}_i]) \\ &= [\overline{x}, \overline{a}_i] \\ &= \textstyle\sum_{l=1}^t (-1)^{p+n_l}[[\overline{x}, \overline{y}_l], \overline{g}_{il}],\end{aligned}$$

where in the last line we have have used (1) translated to Quillen models with appropriate sign change. In the Quillen model for $S^p \times X^m$, $\partial_{S^p\times X^m}([s^p(\overline{y}_l), \overline{g}_{il}]) = [[\overline{x}, \overline{y}_l], \overline{g}_{il}]$ and, for degree reasons, $[s^p(\overline{y}_l), \overline{g}_{il}]$ is the unique element with this property. Thus the element $d_{m-n_j+1}(b_j^*)(\sigma_i) \in \pi_{p+m}(Y_0)$ is the homotopy class represented in the Quillen model of Y by

$$(F_m)_*\left(\sum_{l=1}^t (-1)^{p+n_l}[s^p(\overline{y}_l), \overline{g}_{il}]\right) = (-1)^{p+n_j}[\overline{b}_j, (f_m)_*(\overline{g}_{ij})].$$

The result follows by converting back to Whitehead products. □

4. Consequences and Calculations. We deduce some consequences of Theorem 3.1. The first is a version of Theorem 2.3 for based function spaces. Let $\mathrm{att}(X, \mathbb{Q})$ equal the dimension of the first nontrivially attached cell in a minimal rational CW decomposition for X. (We set $\mathrm{att}(X, \mathbb{Q}) = 0$ if all attaching maps of X are of finite order.) Also let $P_*(Y) = \ker\{\mathrm{ad}_w : \pi_*(X) \to (\mathrm{Der}_w)_*(\pi_*(X))\}$ the kernel of Whitehead adjoint representation. We prove

Theorem 4.1 *Let X be a finite complex with $att(X, \mathbb{Q}) > 0$ and Y a space with $P_n(Y_0) = 0$ for $n \leq att(X, \mathbb{Q})$. If $f : X \to Y$ induces an injection on rational homotopy through degree $att(X, \mathbb{Q})$, then $M_f(X, Y)_* \not\simeq_{\mathbb{Q}} M_0(X, Y)_*$.*

Proof. We show that the rational based Federer spectral sequence does not collapse for f. With notation as in §3, write $[x, a_i]_w = \sum_{j=1}^{t} [[x, y_j]_w, g_{ij}]_w$, for some $g_{ij} \in \pi_{m-n_j+1}(X_m) \otimes \mathbb{Q}$. Since, by hypothesis, $a_i \neq 0$ for some i we know $g_{ij} \neq 0$ for some j. Since f induces an injection on rational homotopy through degree $\mathrm{att}(X, \mathbb{Q})$, $f_*(g_{ij})$ is nontrivial in $\pi_*(Y) \otimes \mathbb{Q}$, as well. Since $P_n(Y_0)$ vanishes for $n \leq \mathrm{att}(X, \mathbb{Q})$ there exists $b_j \in \pi_k(Y) \otimes \mathbb{Q}$ such that $\mathrm{ad}_w(f_*(g_{ij}))(b_j) \neq 0$. Thus by Theorem 3.1, $d_{(m-n_j+1)}(b_j^*)(\sigma_i) = [b_j, f_*(g_{ij})]_w \neq 0$. □

Corollary 4.2 *Let X be a finite complex with $P_n(X_0) = 0$ for $n \leq att(X, \mathbb{Q})$. Then the path components of $M(X, X)_*$ are all of the same rational homotopy type if and only if X is a rational co-H-space.* □

We remark that the hypotheses of Theorem 4.1 and Corollay 4.2 are not superfluous. For example, one proves directly that the components of $M(\mathbb{C}P^n, \mathbb{C}P^m)_*$ are all rationally equivalent.

Theorem 3.1 offers a practical method for computing rational homotopy groups of nontrivial components. To illustrate, we calculate the rational homotopy groups of the space $M_f(X, Y)_*$ when X is a formal space of rational L.S. category two. By the main result of [2], X then occurs as the cofibre of a rationally nontrivial map $h : W_0 \to W_1$ where $W_0 = \bigvee_{i=1}^{s} S^{n_i}$ and

$W_1 = \bigvee_{j=1}^{t} S^{n_j}$. Write $P\widetilde{H}_*(X)$ for the primitive reduced rational homology of X and let $P^c\widetilde{H}_*(X)$ denote a vector space complement in $\widetilde{H}_*(X, \mathbb{Q})$. Then $P\widetilde{H}_*(X) \cong \widetilde{H}_*(W_1)$ and $P^c\widetilde{H}_*(X) \cong s(\widetilde{H}_*(W_0, \mathbb{Q}))$. Let $y_j \in \pi_*(S^{n_j}) \otimes \mathbb{Q}$ be nonzero elements and let $c_1, \ldots, c_t$ be the corresponding basis for $P\widetilde{H}_*(X)$. If $\Delta : X \to X \times X$ is the diagonal map, given $c \in P^c\widetilde{H}_*(X, \mathbb{Q})$ we can write $\Delta_*(c) = 1 \otimes c + c \otimes 1 + \sum_{j,k} q_{jk}(c_j \otimes c_k + (-1)^{n_j n_k} c_k \otimes c_j)$, where $q_{jk} \in \mathbb{Q}$ and $q_{jk} = (-1)^{n_j n_k} q_{kj}$. Given $f : X \to Y$ define

$$D(f)_p : \mathrm{Hom}(P\widetilde{H}_*(X), \pi_{p+*}(Y_0)) \to \mathrm{Hom}(P^c\widetilde{H}_*(X), \pi_{p+*-1}(Y_0))$$

by setting $D(f)_p(b_j^*)(c) = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk}[b_j, f_*(y_k)]_w$. Here, as before, $b_j^* \in \mathrm{Hom}(P\widetilde{H}_*(X, \mathbb{Q}), \pi_{p+*}(Y_0))$ denotes the map which takes the value $b_j \in \pi_{p+n_j}(Y_0)$ on c_j and vanishes on the other c_k. We prove

Theorem 4.3 *Let X be a finite formal complex of rational L.S. category two. Let $f : X \to Y$ be a given map. Then*

$$\pi_p(M_f(X, Y)_*) \otimes \mathbb{Q} \cong \mathit{coker}(D(f)_{p+1}) \oplus \mathit{ker}(D(f)_p).$$

Proof. We identify $\overline{h}_* : \pi_p(M_{f|W_1}(W_1, Y_0)_*) \to \pi_p(M_0(W_0, Y_0)_*)$ with $D(f)_p$. Since X is formal, [3, Proposition 8.8] implies the Quillen model of X is given explicitly as $L_X = F(s^{-1}\widetilde{H}_*(X, \mathbb{Q}))$ with $\partial_X(s^{-1}(c)) = -\sum_{j,k}(-1)^{n_j} q_{jk}[s^{-1}(c_j), s^{-1}(c_k)]$. Let $c \in P^c\widetilde{H}_*(X, \mathbb{Q})$. By linearity, we may suppose that c corresponds to a nonzero element in $\pi_{n_i}(S^{n_i}) \otimes \mathbb{Q}$. If $a_i \in \pi_{n_i}(W_1) \otimes \mathbb{Q}$ represents $h|S^{n_i}$ then by [3, Proposition 8.12] $a_i = \sum_{j,k} q_{jk}[y_j, y_k]_w$. Let $x \in \pi_p(S^p) \otimes \mathbb{Q}$ be nonzero and use the Jacobi identity to obtain

$$\begin{aligned} [x, a_i]_w &= \textstyle\sum_{j,k}[x, [y_j, y_k]_w]_w \\ &= (-1)^{p+1} \textstyle\sum_{j,k} q_{jk} \left([[x, y_j]_w y_k]_w + (-1)^{n_j n_k}[[x, y_k]_w y_j]_w\right). \end{aligned}$$

Thus, in the notation of §3, $g_{ij} = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk} y_k$ and so

$$(\overline{h})_*(b_j^*) = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk}[b_j, f_*(y_k)]_w = D(f)_p(b_j^*). \quad \square$$

Example 4.4. (a) Given $f : \mathbb{C}P^2 \to Y$, we determine the rational homotopy groups of $M_f(\mathbb{C}P^2, Y)_*$. Let $\iota_2 \in \pi_2(\mathbb{C}P^2)$ be nontrivial. By Theorem 4.3 we have

$$\pi_p(M_f(\mathbb{C}P^2, Y)_*) \otimes \mathbb{Q} \cong \operatorname{coker}\{\operatorname{ad}_w f_*(\iota_2) : \pi_{p+1}(Y_0) \to \pi_{p+2}(Y_0)\}$$
$$\oplus \ \ker\{\operatorname{ad}_w f_*(\iota_2) : \pi_p(Y_0) \to \pi_{p+1}(Y_0)\}.$$

(b) Next let $f : S^m \times S^n \to Y$ and let $\iota_m \in \pi_m(S^m)$ and $\iota_n \in \pi_n(S^n)$ be nontrivial. Here Theorem 4.3 implies

$$\pi_p(M_f(S^m \times S^n, Y)_*) \otimes \mathbb{Q} \cong$$

$$\operatorname{coker}\{\operatorname{ad}_w f_*(\iota_n) + \operatorname{ad}_w f_*(\iota_m) : \pi_{p+m+1}(Y_0) \oplus \pi_{p+n+1}(Y_0) \to \pi_{p+m+n}(Y_0)\}$$

$$\oplus \ \ker\{\operatorname{ad}_w f_*(\iota_n) + \operatorname{ad}_w f_*(\iota_m) : \pi_{p+m}(Y_0) \oplus \pi_{p+n}(Y_0) \to \pi_{p+m+n-1}(Y_0)\}.$$

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