Rational Type of Classifying Spaces for Fibrations

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Abstract

We compute the rational cohomology, the rational homotopy Lie algebra and determine the rational homotopy type of the classifying space $Baut_1(X)$ for certain formal spaces X.

1. Introduction.

Let X be a simply connected CW complex of finite type and $aut_1(X)$ the identity component of the space of self-homotopy equivalences of X. The Dold-Lashof classifying space, $Baut_1(X)$, is the classifying space for orientable fibrations with fibre the homotopy type of X [2, 17, 1]. The cohomology of $Baut_1(X)$ thus gives characteristic classes for X-fibrations.

In Appendix 1 of [9], Milnor computed $H^*(Baut_1(S^n), \mathbb{Q})$ to be a polynomial algebra with a single positive degree generator. Thus $Baut_1(S^n)$ is a rational H-space. Milnor's result generalizes to a product of even spheres. In fact, by a result due to Meier [8] and Thomas [19], $Baut_1(X)$ is a rational H-space whenever X is an F_0-space (see §2) whose rational cohomology admits only trivial negative degree derivations (Theorem 1, below).

When X is a product of odd spheres, X is itself a rational H-space. In this case, the rational homotopy type of $Baut_1(X)$ can be deduced from Sullivan's differential graded Lie algebra model for the classifying space [18] (see Theorem 2). To calculate $H^*(Baut_1(X), \mathbb{Q})$ for X a finite mixed product of spheres with no restrictions appears

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quite difficult. (See Example 5.2, below). Our purpose in this paper is to take a step in this direction. We determine the full rational homotopy type of the space $Baut_1(F)$ when F splits as a product of an F_0-space with no negative rational cohomology derivations and a rational H-space, under certain connectivity assumptions. In particular, we compute $H^*(Baut_1(F), \mathbb{Q})$ under these hypotheses and determine precisely when $Baut_1(F)$ is a formal space.

§2. Main Results.

Let $\mathcal{A}$ be a simply connected graded algebra over $\mathbb{Q}$. Define a degree n derivation θ of $\mathcal{A}$ to be a linear self-map satisfying $\theta(\mathcal{A}^k) \subseteq \mathcal{A}^{k-n}$ and $\theta(xy) = \theta(x)y + (-1)^{n|x|}x\theta(y)$. The Lie bracket is the graded commutator: $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2 \circ \theta_1$. Let $Der_+(\mathcal{A})$ denote the graded Lie algebra of all positive degree derivations of $\mathcal{A}$. Given a subalgebra $\mathcal{B}$ of $\mathcal{A}$, let $Der_+(\mathcal{B}, \mathcal{A})$ denote the subalgebra of $Der_+(\mathcal{A})$ consisting of all derivations of $\mathcal{A}$ which vanish on a graded vector space complement of $\mathcal{B}$ in $\mathcal{A}$.

Let L be a connected graded rational Lie algebra. The Lie cohomology of L, $H^*_{\text{Lie}}(L)$, is defined to be cohomology of the differential graded algebra $(\Lambda(sL), d_{[\ ,\ ]})$, where $\Lambda(sL)$ is the free graded algebra generated by the suspension of L and $d_{[\ ,\ ]}$ is dual to the Lie bracket [12, 10, 13]. In [12], Quillen proved that there exists a simply connected space $\|L\|$ satisfying $\pi_*(\Omega\|L\|) \otimes \mathbb{Q} \cong L$ and $H^*(\|L\|, \mathbb{Q}) \cong H^*_{\text{Lie}}(L)$. Specifically, let $(\mathcal{L}, \partial)$ be the minimal Lie algebra model for $(L, 0)$ and take $\|L\|$ to be the spatial realization of $(\mathcal{L}, \partial)$. By construction, $\|L\|$ is a *coformal* space; that is, the rational homotopy type of $\|L\|$ is a formal consequence of its rational homotopy Lie algebra.

A simply connected space X is an F_0-*space* if X has finite-dimensional rational homotopy and cohomology with

$$dim\pi_{even}(X) \otimes \mathbb{Q} = dim\pi_{odd}(X) \otimes \mathbb{Q} \quad \text{and} \quad H^{odd}(X, \mathbb{Q}) = 0.$$

The class includes (products of) spheres, complex projective spaces and homogeneous spaces G/H with rank$G$ = rank$H$. In [5], Halperin conjectured $Der_+(H^*(X, \mathbb{Q})) = 0$ for all F_0-spaces X. This condition is equivalent to the collapsing of the rational Serre spectral sequence for all orientable fibrations with fibre X [8]. Halperin's conjecture has been confirmed in many special cases [6, 14]; in particular, for the examples mentioned above.

Given a graded vector space Z, let $min(Z) = min\{n | Z^n \neq 0\}$ and

$max(Z) = max\{n | Z^n \neq 0\}$. We prove the following results on the rational homotopy type of classifying spaces in Section 3:

Theorem 1 *Let X be an F_0-space with $Der_+(H^*(X,\mathbb{Q})) = 0$. Define integers d_{2n} for $n = 1, 2, \ldots,$ by*

$$d_{2n} = \sum_{k \geq n} dim(H^{2k-2n}(X,\mathbb{Q}))\left[dim(\pi_{2k-1}(X) \otimes \mathbb{Q}) - dim(\pi_{2k}(X) \otimes \mathbb{Q})\right].$$

Then

$$Baut_1(X) \simeq_{\mathbb{Q}} \prod_n K(\mathbb{Q}^{d_{2n}}, 2n).$$

Corollary 1 *Let X be as in Theorem 1. Let $\mathcal{L}_1(X)$ denote the oddly graded rational vector space having dimension d_{2n} in degree $2n-1$. View $\mathcal{L}_1(X)$ as an abelian Lie algebra. Then $Baut_1(X)$ is formal and coformal with*

$$\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q} \cong \mathcal{L}_1(X) \quad \text{and} \quad H^*(Baut_1(X), \mathbb{Q}) \cong \Lambda(s\mathcal{L}_1(X)).$$

Theorem 2 *Let Y be a simply connected rational H-space of finite type. Then $Baut_1(Y) \simeq_{\mathbb{Q}} \|Der_+(H^*(Y,\mathbb{Q}))\|$.*

Corollary 2 *Let Y be as in Theorem 2. Then $Baut_1(Y)$ is coformal with*

$$\pi_*(\Omega Baut_1(Y)) \simeq_{\mathbb{Q}} Der_+(H^*(Y,\mathbb{Q})) \quad \text{and}$$
$$H^*(Baut_1(Y), \mathbb{Q}) \cong H^*_{\mathrm{Lie}}(Der_+(H^*(Y,\mathbb{Q})).$$

Theorem 3 *Let $F = X \times Y$ where X and Y are as in Theorems 1 and 2, respectively. Suppose that*

$$min(\pi_*(X) \otimes \mathbb{Q}) + min(\pi_*(Y) \otimes \mathbb{Q}) \geq max(\pi_*(F) \otimes \mathbb{Q}) \quad \text{and}$$
$$max(\pi_*(X) \otimes \mathbb{Q}) \leq min(\pi_*(Y) \otimes \mathbb{Q}).$$

Then

$$Baut_1(F) \simeq_{\mathbb{Q}} Baut_1(X) \times \|Der_+(H^*(Y,\mathbb{Q}), H^*(F,\mathbb{Q}))\|.$$

Corollary 3 *Let F be as in Theorem 3. Then $Baut_1(F)$ is a coformal space with*

$$\pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q} \cong \mathcal{L}_1(X) \times Der_+(H^*(Y,\mathbb{Q}), H^*(F,\mathbb{Q})) \quad \text{and}$$

$$H^*(Baut_1(F), \mathbb{Q}) \cong \Lambda(s\mathcal{L}_1(X)) \otimes H^*_{\mathrm{Lie}}(Der_+(H^*(Y,\mathbb{Q}), H^*(F,\mathbb{Q}))).$$

In Section 4, we study the formality of spatial realizations of finite-dimensional graded Lie algebras. We obtain:

Theorem 4 *Let Y be a simply connected rational H-space with finite-dimensional rational homotopy. Then $Baut_1(Y)$ is formal if and only if*

i) $\pi_*(Y) \otimes \mathbb{Q}$ is concentrated in a single degree or

ii) $\pi_*(Y) \otimes \mathbb{Q}$ is concentrated in two consecutive odd degrees $2n-1, 2n+1$ with $dim\pi_{2n-1}(Y) \otimes \mathbb{Q} \leq dim\pi_{2n+1}(Y) \otimes \mathbb{Q}$ or

iii) $\pi_*(Y) \otimes \mathbb{Q}$ is concentrated in two odd degrees $p < q$ with $dim\pi_p(Y) \otimes \mathbb{Q} = 1$.

Theorem 5 *Let $F = X \times Y$ be as in Theorem 3. Then $Baut_1(F)$ is formal if and only if Y satisfies condition i), ii) or iii) above.*

§3. Rational Homotopy of Classifying Spaces.

A connected differential graded Lie algebra (*dgla*) (L, ∂) is a *model* for a simply connected complex X if the spatial realization of the Quillen minimal model for (L, ∂) is rationally equivalent to X. In this case, in particular, $H(L, \partial)$ is the rational homotopy Lie algebra of X. The following result is just a restatement of the definition of coformality:

Lemma 3.1 *Let X be a simply connected complex. Then X is coformal if and only if X has a dgla model (L, ∂) with $\partial = 0$.* □

In [18], Sullivan describes a dgla model for $Baut_1(X)$ when X is a simply connected complex of finite type. Let $(\mathcal{M}_X, d_X)$ be the Sullivan minimal model for X. Define a degree -1 derivation ∂_X of $Der_+(\mathcal{M}_X)$ by $\partial_X(\theta) = [d_X, \theta]$. Define a subalgebra $\widetilde{Der}_+(\mathcal{M}_X)$ of $Der_+(\mathcal{M}_X)$ by restricting, in degree 1, to derivations of $\mathcal{M}_X$ which vanish under ∂_X. The pair $(\widetilde{Der}_+(\mathcal{M}_X), \partial_X)$ is a dgla model for $Baut_1(X)$. For an indirect proof, use [4, Theorem 2] and the proof for the Schlessinger-Stasheff model [13].

Proof of Theorem 2. Since Y is a rational H-space, its Sullivan minimal model is just $(H^*(Y, \mathbb{Q}), 0)$. In the Sullivan dgla model for $Baut_1(Y)$, $\partial_X = 0$. The result follows from Lemma 3.1. □

We next recall some notation and results from [16]. We say a simply connected space X is *two-stage* if its Sullivan minimal model $(\mathcal{M}_X, d_X)$ can be written $\mathcal{M}_X = \Lambda(V_0) \otimes \Lambda(V_1)$, for finite-dimensional graded vector spaces V_0 and V_1 with $d_X(V_0) = 0$ and $d_X(V_1) \subseteq$

$\Lambda(V_0)$. We may assume $d_X : V_1 \to \Lambda(V_0)$ is an injection. Write $V_0 = \mathbb{Q}(x_1, \ldots, x_m)$ and $V_1 = \mathbb{Q}(y_1, \ldots, y_n)$ so that $d_X(x_i) = 0$ and $d_X(y_j) = R_j(x_1, \ldots, x_m)$, a nontrivial polynomial without linear term in the x_i. By [3], X is formal if and only if the sequence $R_1, \ldots, R_n$ is regular in $\Lambda(V_0)$.

Define graded spaces $L_0(X)$ and $L_1(X)$ by setting

$$L_i^n(X) = \bigoplus_{k \geq 0} H^k(X, \mathbb{Q}) \otimes V_i^{n+k},$$

where $n > 0$ for $i = 0$ and $n \geq 0$ for $i = 1$. Let $L(X) = L_0(X) \oplus L_1(X)$. Thus $L(X)$ is spanned by elements $\alpha \otimes z_k$ where $z_k \in V_0 \oplus V_1$ is one of our basis vectors and $\alpha \in H^*(X, \mathbb{Q})$ is homogeneous of degree no more than $|z_k|$. The degree of $\alpha \otimes z_k$ is the difference $|z_k| - |\alpha|$.

Define a linear map $D : L_0(X) \to L_1(X)$ of degree -1 by

$$D(\alpha \otimes x_i) = \sum_{j=1}^{n} \alpha \cdot \left\{ \frac{\partial R_j}{\partial x_i} \right\} \otimes y_j,$$

where $\{P\}$ denotes the cohomology class in $H^*(X, \mathbb{Q})$ represented by an element $P \in \mathcal{M}_X$. Let

$$\mathcal{L}_0(X) = ker\{D : L_0(X) \to L_1(X)\} \quad \text{and}$$

$$\mathcal{L}_1(X) = cok\{D : L_0(X) \to L_1(X)\},$$

where $\mathcal{L}_1(X)$ is made connected by eliminating its elements of degree zero. By [16, Theorem 3.2], there is an isomorphism of graded vector spaces

$$\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q} \cong \mathcal{L}_0(X) \oplus \mathcal{L}_1(X)$$

for any two-stage space X.

Proof of Theorem 1. Let X be an F_0-space. Then, in the notation above, V_0 is evenly graded, V_1 is oddly graded and $\dim V_0 = \dim V_1$. Since $H^*(X, \mathbb{Q})$ is evenly graded, $L_0(X)$ is evenly graded and $L_1(X)$ is oddly graded. By hypothesis, X satisfies Halperin's conjecture. Meier [8] and Thomas [19] independently proved that Halperin's conjecture is true for an F_0-space X if and only if the space $Baut_1(X)$ has evenly graded rational homotopy groups. Thus, $\mathcal{L}_0(X) = 0$. The result now follows from the observation that any space with evenly graded rational homotopy is a rational H-space. $\square$

Proof of Theorem 3. Let W be a finite-dimensional graded rational vector space such that $H^*(Y, \mathbb{Q}) \cong \Lambda(W)$. Write $W = \mathbb{Q}(w_1, \ldots w_k)$.

Repeat the above constuction for F. The Sullivan minimal model $(\mathcal{M}_F, d_F)$ is $\mathcal{M}_F = \Lambda(Z_0) \otimes \Lambda(Z_1)$ where $Z_0 = W \oplus V_0,\ Z_1 = V_1$, $d_F(Z_0) = 0$ and $d_F|Z_1 = d_X|V_1$. Computing the image of $D : L_0(F) \to L_1(F)$ directly, gives

$$\mathcal{L}_1(F) \cong \mathcal{L}_1(X) \oplus \bigoplus_{n>0} \bigoplus_{k>0} H^k(Y, \mathbb{Q}) \otimes V_1^{n+k}.$$

However, by our second degree hypothesis, $V_1^m = 0$ for $m \geq min\{W\}$. Thus $\mathcal{L}_1(F) = \mathcal{L}_1(X)$.

For the kernel of D, [16, Theorem 3.4] implies

$$\mathcal{L}_0(F) = \bigoplus_{n>0} \bigoplus_{k\geq 0} H^k(F, \mathbb{Q}) \otimes W^{n+k} \cong Der_+(H^*(Y, \mathbb{Q}), H^*(F, \mathbb{Q})),$$

as graded vector space. Using [16, Theorem 3.2] again, we conclude

$$\text{(1)} \quad \pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q} \cong \mathcal{L}_1(X) \oplus Der_+(H^*(Y, \mathbb{Q}), H^*(F, \mathbb{Q})),$$

as graded vector space.

In §4 of [16], we identify cycle representatives in Sullivan's dgla model for the vector subspaces $\mathcal{L}_i(F)$, whenever F is formal and two-stage. Define an *elementary derivation* $P\partial z_k$ in $\widetilde{Der}_+(\mathcal{M}_F)$ to be the derivation which carries the basis element z_k of $Z_0 \oplus Z_1$ to $P \in \mathcal{M}_F$ and vanishes on the other basis elements. Here $|P| < |z_k|$ and, if the difference is 1, we need to restrict to the kernel of ∂_F. Define a linear surjection $p : \widetilde{Der}_+(\mathcal{M}_F) \to L(F)$ by $p(P\partial z_k) = \rho(P) \otimes z_k$ where $\rho : (\mathcal{M}_F, d_F) \to (H^*(F, \mathbb{Q}), 0)$ is a formalization for F. Define subspaces $D_0(F), D_1(F)$ and $B(F)$ of $\widetilde{Der}_+(\mathcal{M}_F)$ by

$$\begin{aligned} D_0(F) &= \text{Span}\{P\partial z_i \ \mid \ P \in \Lambda(Z_0), |P| < |z_i|, \ \ z_i \in Z_0\} \\ D_1(F) &= \text{Span}\{P\partial z_j \ \mid \ P \in \Lambda(V_0), |P| < |z_j|, \ \ z_j \in Z_1\} \quad \text{and} \\ B(F) &= \text{Span}\{P z_k \partial z_j | \ \ P \in \Lambda(Z_0), |P| + |z_k| < |z_j|, \ \ z_j, z_k \in Z_1\}. \end{aligned}$$

Note $\partial_F(D_1(F)) = 0$ and $\partial_F(B(F) \oplus D_0(F)) \subseteq D_1(F)$. Set

$$\mathcal{D}_0(F) = ker\{\partial_F : B(F) \oplus D_0(F) \to D_1(F)\} \ \text{ and}$$

$$\mathcal{D}_1(F) = cok\{\partial_F : B(F) \oplus D_0(F) \to D_1(F)\}.$$

We view $\mathcal{D}_1(F)$ as a subspace of $\text{Der}_+(\mathcal{M}_F)$ by taking vector space complements in each degree.

By [16, Theorem 4.2], p induces an isomorphism $p : \mathcal{D}_1(F) \rightarrow \mathcal{L}_1(F)$ and a surjection $p : \mathcal{D}_0(F) \rightarrow \mathcal{L}_0(F)$ both of which correspond to the homology projection in $(\widetilde{Der}_+(\mathcal{M}_F), \partial_F)$. We use this result to complete the proof.

Our second degree hypothesis implies $\mathcal{D}_1(F) \cong \mathcal{L}_1(X)$, as above. Choose an additive space Z of cocycle representatives in $\mathcal{M}_X$ for $\widetilde{H}^*(X, \mathbb{Q})$. Define a subspace $\mathcal{D}_0'(F)$ of $\mathcal{D}_0(F)$ by

$$\mathcal{D}_0'(F) = \mathrm{Span}\{P\partial w_k \ \mid \ P \in Z \oplus W, |P| < |w_k|, \quad w_k \in W\}.$$

Our first degree hypothesis implies decomposables in $H^*(F, \mathbb{Q})$ involving elements from both $\widetilde{H}^*(X, \mathbb{Q})$ and $\widetilde{H}^*(Y, \mathbb{Q})$ occur in degrees above $\geq max(W)$. It follows that

$$\mathcal{D}_0'(F) \cong Der_+(H^*(Y, \mathbb{Q}), H^*(F, \mathbb{Q})),$$

as graded Lie algebras.

Finally, note the subspace $\mathcal{D}_1(F) \oplus \mathcal{D}_0'(F)$ of cycles of $\widetilde{Der}_+(\mathcal{M}_F)$, is actually a subalgebra. Moreover, $\mathcal{D}_1(F)$ consists of indecomposable central elements of $\mathcal{D}_1(F) \oplus \mathcal{D}_0'(F)$. Thus we may write this subalgebra as $\mathcal{D}_1(F) \times \mathcal{D}_0'(F)$. By [16, Theorem 4.2] and Equation 1 above, the inclusion $(\mathcal{D}_1(F) \times \mathcal{D}_0'(F), 0) \hookrightarrow (\widetilde{Der}_+(\mathcal{M}_F), \partial_F)$, induces an equivalence on homology. Thus $(\mathcal{D}_1(F) \times \mathcal{D}_0'(F), 0)$ is a dgla model for $Baut_1(F)$. The result follows from Lemma 3.1. □

4. Formality of Spatial Realizations of Graded Lie Algebras.

Given a connected graded Lie algebra L over $\mathbb{Q}$, let $nil(L)$ denote the nilpotency of L. That is, $nil(L)$ is the length of the longest iterated bracket in L. We prove

Theorem 4.1 *Let L be a finite-dimensional connected graded Lie algebra over $\mathbb{Q}$. If $nil(L) > 2$ then $\|L\|$ is not formal.*

Proof. The Sullivan minimal model for $\|L\|$ is $(\Lambda(sL), d_{[\ ,\ ]})$ where $d_{[\ ,\ ]}$ is dual to the bracket in L [11, Proposition 3.3]. Suppose $\|L\|$ is formal. Then $\|L\|$ is formal with finite-dimensional rational homotopy and so hyperformal [3, Theorem 1]. In particular, $(\Lambda(sL), d_{[\ ,\ ]})$ is a two-stage dga which we may assume is in "normal form". That is, by [15, Lemma 3.3], we may assume $sL = V_0 \oplus V_1$ where $d_{[\ ,\ ]}(V_0) = 0$, $d_{[\ ,\ ]}(V_1) \subseteq \Lambda(V_0)$, $d_{[\ ,\ ]}|V_1$ is an injection and $d_{[\ ,\ ]}(sw) \neq d_{[\ ,\ ]}(\alpha)$ for any $sw \in V_1$ and decomposable $\alpha \in \Lambda(sL)$.

Choose $sw \in V_1$ homogeneous of minimal positive degree with $w = [[x, y], z]$ for $x, y, z \in L$. Since $d_{[\ ,\ ]}(V_0) = 0$ we may assume

$v = [x,y] \in s^{-1}V_1$ and $z \in s^{-1}V_0$. Let $n = |w|$ and consider the nth Postnikov section, X_n, of $\|L\|$. The minimal model for X_n is $(\Lambda((sL)^{\leq n}), d_{[\ ,\ ]})$. Using the normal form above, we see that the element $d_{[\ ,\ ]}(sw)$ is a cocycle of $(\Lambda((sL)^{\leq n}), d_{[\ ,\ ]})$ which does not bound. Moreover, the element $sv \cdot sz$ is a nontrivial summand of this cocycle which does not bound. Thus $d_{[\ ,\ ]}(sw)$ represents an indecomposable element in $H^{n+1}(X_n, \mathbb{Q})$. By [15, Theorem 4.6], this contradicts the hyperformality of $\|L\|$. $\square$

Lemma 4.2 *Let $F = \Lambda(a_i, b_{ij})$ be the free graded algebra generated by elements a_i of degree k and b_{ij} of degree l for $l \neq k$, $i = 1, \ldots n$, $j = 1, \ldots, m$. Set $R_j = \sum_{i=1}^n a_i b_{ij}$ for $j = 1, \ldots, m$. Then $R_1, \ldots, R_m$ is a regular sequence of F if and only if k and l are both even and $m \leq n$.*

Proof. If either k or l is odd then each R_j is a zero-divisor in F. If k and l are both even, consider the $m \times n$ matrix $B = (b_{ij})$, whose jth row of B represents R_j in the basis $\{a_1, \ldots, a_n\}$. If $m > n$ we can row reduce B to find a nontrivial linear combination $\beta_1 R_1 + \cdots + \beta_m R_m = 0$, where $\beta_j \in \Lambda(b_{ij})$. Thus the sequence is not regular when $m > n$.

Conversely, suppose k and l are even and $m \leq n$. If R_m is a zero divisor in $F/(R_1, \ldots, R_{m-1})$ then $\alpha R_m = \alpha_1 R_1 + \cdots + \alpha_{m-1} R_{m-1}$ for some $\alpha \notin (R_1, \ldots, R_{m-1})$. But this expression implies a nontrivial solution to the homogeneous system $BX = 0$ which is impossible. Since permutations of regular sequences are regular in a polynomial algebra, the regularity of the sequence follows by induction. $\square$

Proof of Theorem 4. By Theorem 1, $Baut_1(Y) \simeq_{\mathbb{Q}} \|Der_+(H^*(Y, \mathbb{Q}))\|$. By hypothesis, $H^*(Y, \mathbb{Q})$ is a finitely generated free graded algebra. Thus $Der_+(H^*(Y, \mathbb{Q}))$ is a finite-dimensional graded connected Lie algebra. By [16, Theorem 5.3] (or direct calculation), $nil(Der_+(H^*(Y, \mathbb{Q})) \geq 2$ unless $\pi_*(Y) \otimes \mathbb{Q}$ is concentrated in either a single degree or in two degrees $k < l$ with $l - k \leq 2$ or k odd and $dim \pi_k(Y) \otimes \mathbb{Q} = 1$. In the former case, $Der_+(H^*(Y, \mathbb{Q}))$ is abelian and so $Baut_1(Y)$ is a rational H-space.

In the latter cases, write $H^*(Y, \mathbb{Q}) = \Lambda(w_1, \ldots, w_n, v_1, \ldots, v_m)$ where $|w_i| = k$ and $|v_j| = l$. Let $\alpha_i = 1 \partial w_i, \beta_{ij} = w_i \partial v_j$ and $\gamma_j = 1 \partial v_j$ Then $Der_+(H^*(Y, \mathbb{Q})) = \mathbb{Q}(\alpha_i, \beta_{ij}, \gamma_j)$ with $\gamma_j = [\alpha_i, \beta_{ij}]$, the only nontrivial brackets. Thus letting $a_i = s\alpha_i, b_{ij} = s\beta_{ij}$ and $c_j = s\gamma_j$, the Sullivan model for $Baut_1(Y)$ is just $(\Lambda(a_i, b_{ij}, c_j), d_{[\ ,\ ]})$ where $d_{[\ ,\ ]}(a_i) = d_{[\ ,\ ]}(b_{ij}) = 0$ and $d_{[\ ,\ ]}(c_j) = 1/n R_j$. Since $Baut_1(Y)$ has finite-dimensional rational homotopy, $Baut_1(Y)$ is formal if and only if the sequence $R_1, \ldots, R_m$ is regular in F ([3, Theorem 1] again).

The result thus follows from Lemma 4.2. □

Proof of Theorem 5. By [16, Theorem 5.3],

$$nil(\pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q}) = nil(\pi_*(\Omega Baut_1(Y)) \otimes \mathbb{Q})$$

when $F = X \times Y$, as hypothesized. Thus we are again reduced to the cases where $\pi_*(Y) \otimes \mathbb{Q}$ is concentrated in one or two degrees, as above. In the latter case, we choose an additive homogenous basis $\{z_1, \ldots, z_d\}$ for $H^*(X, \mathbb{Q})$ and set $\alpha_{ik} = z_k \partial w_i, \beta_{ij} = w_i \partial v_j$ and $\gamma_k = z_k \partial v_j$. Then $Der_+(H^*(Y, \mathbb{Q}), H^*(F, \mathbb{Q})) = \mathbb{Q}(\alpha_{ik}, \beta_{ij}, \gamma_{jk})$ with $\gamma_{jk} = [\alpha_{ik}, \beta_{ij}]$, the only nontrivial brackets. The Sullivan model for $Baut_1(F)$ is thus of the form $(\Lambda(s\mathcal{L}_1(X)), 0) \otimes (\Lambda(a_{ik}, b_{ij}, c_{jk}), d_{[\ ,\ ]})$ with $d_{[\ ,\ ]}(a_{ik}) = d_{[\ ,\ ]}(b_{ij}) = 0$ and $d_{[\ ,\ ]}(c_j) = 1/nR_{jk}$, where $R_{jk} = \sum_{i=1}^n a_{ik} b_{jk}$. The result follows, as above, from Lemma 4.2. □

5. Applications and Examples.

The following are consequences of our results and the proof of special cases of the Halperin conjecture.

5.1 Applications.

(1) Let $F = S^{2m_1} \times \cdots \times S^{2m_k} \times S^{2n_1+1} \times \cdots \times S^{2n_l+1}$, where $m_1 \leq \cdots \leq m_k, n_1 \leq \cdots \leq n_l$. If $l = 0$ then $Baut_1(F)$ is a rational H-space. If $k = 0$ then $Baut_1(F)$ is a coformal space which is formal if and only if $i)$ all the n_j are equal, $ii)$ $n_j < n_{j+1}$ for a unique $j, n_j + 1 = n_{j+1}$ and $j \leq l - j$, or $iii)$ $n_1 < n_2 = \cdots = n_l$.

When both $k, l > 0$, Theorem 3 applies provided $2m_k \leq n_1 + 1$ and $m_1 + n_1 \geq 2n_l + 1$. In this case, $Baut_1(F)$ is a coformal space having the rational H-space $Baut_1(S^{2m_1} \times \cdots \times S^{2m_k})$ as a rational factor. Moreover, $Baut_1(F)$ is formal if and only if one of conditions $i), ii)$ or $iii)$ is satisfied. This example generalizes directly to spaces F whose rational cohomology is a tensor product of free and truncated polynomial algebras. □

(2) Let X be an F_0-space with $Der_+(H^*(X, \mathbb{Q})) = 0$. Let $q \geq max(\pi_*(X) \otimes \mathbb{Q})$. Then $Baut_1(X \times K(V, q))$ is a rational H-space for any finite-dimensional rational vector space V. In particular, let $F = U(n)/U(n_1) \times \cdots \times U(n_k)$ or $F = Sp(n)/Sp(n_1) \times \cdots Sp(n_k)$ where $n - 1 \leq n_1 + \cdots + n_k \leq n$. Then $Baut_1(F)$ is a rational H-space. □

(3) If q is odd and W is another rational vector space with $dimW \geq dimV$, the space $Baut_1(X \times K(V, q) \times K(W, q + 2))$ is both coformal and formal. In particular, if $F = U(n)/U(n_1) \times \cdots \times U(n_k)$ and $n - 2 \leq n_1 + \cdots n_k \leq n$ then $Baut_1(F)$ is both formal and coformal. □

Finally, we show the necessity of the hypotheses in Theorem 3.

5.2 Examples

(1) Let $F = S^2 \times S^3 \times S^7$, so that F violates the first but not the second degree hypothesis in Theorem 3. We show $Baut_1(S^2)$ is not a rational factor of $Baut_1(F)$. This implies, in particular, that $Baut_1(F)$ is not coformal since $\pi_*(\Omega Baut_1(S^2)) \otimes \mathbb{Q}$ is a factor of $\pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q}$ by [16, Theorem 5.3].

Note $Baut_1(S^2) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4)$. In the notation of §3, $V_0 = \mathbb{Q}(x_1), W = \mathbb{Q}(x_2, x_3)$ and $V_1 = \mathbb{Q}(y)$ where $|x_1| = 2, |x_2| = 3, |x_3| = 7, |y| = 3$ and $d_F(y) = x_1^2$. Let $\mathcal{D}$ denote the subalgebra of $\widetilde{Der}_+(\mathcal{M}_F)$ spanned by:

$$a = x_1\partial x_2, \quad b = x_1x_2\partial x_3, \quad c_1 = 1\partial y, \quad c_2 = 1\partial x_2, \quad c_3 = x_1^2\partial x_3$$

$$d_1 = y\partial x_3, \quad d_2 = x_2\partial x_3, \quad e = x_1\partial x_3, \quad f = 1\partial x_3.$$

The only boundary is $\partial_X(d_1) = -c_3 = -[a, b]$. Thus $\mathcal{D}$ is, in fact, a sub-dgla of $(\widetilde{Der}_+(\mathcal{M}_F), \partial_F)$. By [16, Theorems 3.1 and 4.2], $H(\mathcal{D}, \partial_X) \cong \pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q}$. Thus the inclusion $(\mathcal{D}, \partial_X) \hookrightarrow (\widetilde{Der}_+(\mathcal{M}_F), \partial_F)$ induces a homology isomorphism: $(\mathcal{D}, \partial_X)$ is a dgla model for $Baut_1(F)$. Note that $[c_1, d_1] = [c_2, d_2] = f$. Let $\xi : (\mathcal{L}, \partial) \to (\mathcal{D}, \partial_X)$ denote the Quillen model for $(\mathcal{D}, \partial_X)$. For $i = 1, 2$, let $\bar{c_i}, \bar{d_i}$ in $\mathcal{L}$ denote the preimages of the indecomposables $c_i, d_i \in \mathcal{D}$. Then there exists an indecomposable $\bar{g} \in \mathcal{L}$ with $\partial(\bar{g}) = [\bar{c_1}, \bar{d_1}] - [\bar{c_2}, \bar{d_2}]$. By [11, Proposition 2.4], $H^4(Baut_1(F), \mathbb{Q}) = \mathbb{Q}(s\bar{c_1}, s\bar{c_2})$ and $s\bar{c_1} \cdot s\bar{d_1} = s\bar{c_2} \cdot s\bar{d_2} \in H^9(Baut_1(F), \mathbb{Q})$. Thus, since $H^*(Baut_1(F), \mathbb{Q})$ has no free factor generated in degree four, $Baut_1(S^2)$ is not a rational factor of $Baut_1(F)$. $\square$

(2) Let $F = S^3 \times S^4 \times S^7$, so that F violates the second but not the first degree hypothesis in Theorem 3. In this case, it is easy to see that $Baut_1(S^4)$ is not a rational factor of $Baut_1(F)$. For $\pi_*(\Omega Baut_1(S^4)) \otimes \mathbb{Q} = \mathbb{Q}(1 \otimes y)$ where $y \in \pi_7(S^4) \otimes \mathbb{Q}$ is nontrivial. Using cycle representatives in Sullivan's model ([16, Theorem 4.2]), we have $1 \otimes y = q[1 \otimes x, \alpha \otimes y]$ in $\pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q}$, where $x \in \pi_3(S^3) \otimes \mathbb{Q}$ and $\alpha \in H^3(S^3, \mathbb{Q})$ are nontrivial. Thus $\pi_*(\Omega Baut_1(S^4)) \otimes \mathbb{Q}$ is not a factor of $\pi_*(\Omega Baut_1(F)) \otimes \mathbb{Q}$. $\square$

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